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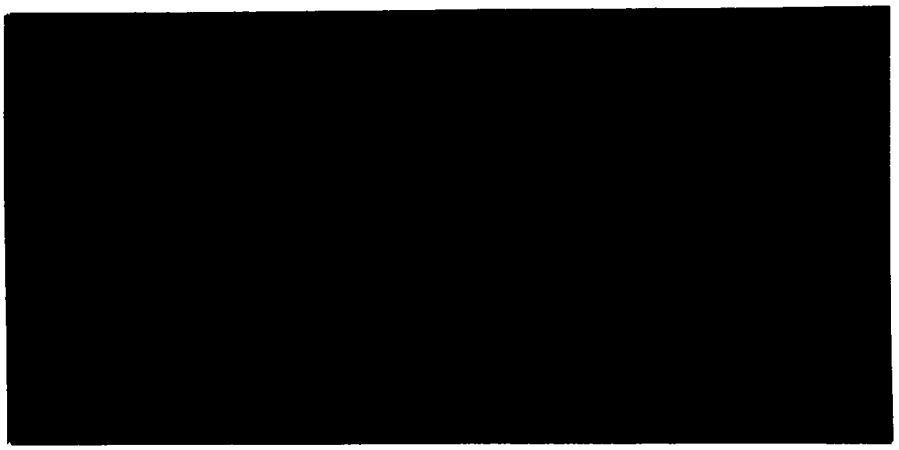

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# THE NUMERICAL TREATMENT OF SIMPLE HYDRODYNAMIC SHOCKS USING THE VON NEUMANN-RICHTMYER METHOD

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PHYSICS

ABSTRACT

In most cases, when one attempts to solve the partial differential equations describing the flow of a compressible fluid, it is found that (1) either it is impossible to solve the equations analytically, or (2) an analytic solution can be found for special cases only. An approximation to a solution can often be found, however, by the use of numerical procedures. Of course the questions of uniqueness of a solution and whether or not the solution obtained by numerical methods actually approximates a solution of the differential equations always arise, but in most cases one knows pretty well what general form of the solution to expect, and if the solution assumes this form it is usually accepted.

The validity of the method of approximating solutions to differential equations by numerical procedures can be tested by applying the method to equations which have known solutions for the boundary and initial conditions given. For example, the differential equations of compressible fluid motion have a unique solution over a certain region when there is a single steady shock moving through the region. The shock can be thought of as a surface of discontinuity in the hydrodynamic functions across which the values of the functions are connected by certain shock conditions. Another way to treat shocks is the method of Von Neumann and Richtmyer, which introduces frictional forces into the equations by use of pseudo-viscosity term. The latter method is much the simpler of the two when numerical procedures are used, since

the equations describe the flow in shock regions as well as at other points in the fluid, so the difficult application of the shock conditions is not necessary.

The actual numerical integration of the equations is accomplished by replacing the derivatives with finite differences, then solving the difference equations resulting from this substitution by step-by-step methods. Such a numerical calculation would take months if done by hand, so such a calculation would ordinarily not be attempted unless one had at his disposal a large scale, automatic digital calculating machine, which would reduce the problem to one taking only a few minutes.

The results of this endeavor verify the fact that numerical integration of the equations gives quite a good approximation to the actual solution; indeed, the agreement between the analytical functions and the functions obtained by numerical methods is less than one percent. This gives us confidence in the method when it is applied to more complicated flows for which it is impossible to find an analytic solution.

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# CHAPTER I

## INTRODUCTION

When the differential equations describing the flow of compressible fluids are derived under the assumptions that (1) forces in the fluid are due only to variations in pressure and (2) that the entropy of any volume element remains constant, it can be shown mathematically that these differential equations cannot have continuous solutions under all circumstances. If we adopt the notion of "shock discontinuities" in these solutions, the differential equations describing the flow in continuous regions together with conditions expressing the laws of conservation across the discontinuities suffice to completely determine the flow. An alternative procedure is to use a method, developed recently by J. Von Neumann and R. D. Richtmyer [3] , which introduces frictional forces by way of inclusion of a pseudo-viscosity term in the differential equations. When the latter method is used, the discontinuities no longer exist. Instead, we have narrow regions over which each hydrodynamic function assumes the form of a continuous curve resembling a portion of a sine wave. The Von Neumann - Richtmyer method is much simpler to use when solving the equations by numerical procedures. In the application of numerical methods, the former method becomes almost impossible to use for complicated flows with more than one discontinuity, since actual applications of the conditions across these discontinuities is quite difficult and lengthy.

In Chapter II, the differential equations of compressible fluid flow are developed completely, under the assumption that variations in pressure are the only forces in the fluid and that the entropy of any element of mass of the fluid remains constant. In addition, the equation of state of an ideal gas is discussed in some detail, and both the Lagrangian and Eulerian viewpoints of describing the flow, along with the method of transforming from one system to another, are discussed.

Chapter III shows the effect of both small disturbances (sound waves) and large disturbances (which develop into shocks) on the fluid flow. The jump conditions expressing the conservation laws across shocks, and many important formulas following directly from these, are derived. Analytic solutions to the flow problem resulting when a piston is pushed into a tube containing ideal gas are obtained.

Chapter IV discusses the Von Neumann - Richtmyer method. In this chapter the procedure involved in "smearing out" the shocks is demonstrated, and the resulting shape and width of the smeared shocks is discussed.

In Chapter V, difference equations corresponding to the Von Neumann - Richtmyer flow are formulated. The solution to these difference equations, with certain boundary and initial conditions imposed, obtained by numerical methods, are seen to agree quite well with the analytic solution to the differential equations with the same conditions imposed.

Interest in obtaining approximate solutions to problems governed by

differential equations by using numerical methods has been increasing in recent years. The validity of these methods has been discussed many times and in this paper this validity is demonstrated heuristically by comparing the solution to the differential equations and the solution to the difference equations, where both sets of equations have the same conditions imposed on them.

To obtain the solutions to the difference equations, a large-scale digital calculating machine, the IBM Type 701 EDPM, was used. The results of this endeavor are shown in Chapter IV in the form of tables and graphs.

## CHAPTER II

## MOTION OF IDEAL GAS

Fluids are distinguished from solids by the fact that particles of fluid are readily displaced. Indeed, if a fluid is in equilibrium, it offers no resistance whatever to a gradual change of shape. However, when the equilibrium is disturbed, i.e., a rapid change of shape is attempted, a fluid will offer resistance to this change, but this resistance disappears rapidly once the deforming motion is ceased.

If the fluid is in motion, the equations governing this motion are the mathematical expressions of the laws of conservation of mass, momentum, and energy. In addition an expression called the equation of state which relates the thermodynamic variables is required.

Gases are fluids with the following properties: by a suitable application of forces they can be compressed into a very small space; and if they are given more space than they occupy in their original condition, they will eventually fill the space uniformly with a decrease in the force per unit area (which from here on will be referred to simply as "pressure") exerted by the gas. Because of this property gases are thermodynamically synonymous with compressible fluids, and it is with this type of fluid that the following discussion is concerned. Incompressible fluids are those having constant volume under all conditions. Incompressible fluids are called liquids by thermodynamicists.

The equations describing fluid motion are usually written in one of two coordinate systems; one of these is attributed to Lagrange, the

other to Euler. Lagrange's system describes the motion in terms of the paths of individual particles of the fluid. In Euler's representation, attention is directed to fixed points in space and to what happens at these points in time. A simple physical example serves best to illustrate the difference in the two methods. Suppose one desires to observe the flow of air on a windy day. A balloon filled with enough lighter-than-air gas to make it weightless could be released in the wind; observation of the path of this object would give a Lagrangian interpretation of the flow. Another way to observe the flow of air is to place a device to measure wind velocity at a fixed point. This would give an Eulerian interpretation of the motion.

Mathematically, if we consider the motion in one dimension only, in the Lagrangian system the coordinate X of the particle is a function of the time t and the parameter which characterizes the particle; this parameter will be chosen, in our work, as the co-ordinate of the particle at t = 0. In the Eulerian scheme the mathematical representation is effected by giving as a function of position x and time t the velocity u of the particle that happens to be at x at time t.

Suppose it becomes desirable to shift from one system to another, say from the Lagrangian system to the Eulerian. Equating the Lagrangian and Eulerian co-ordinates gives

(see Fig. 1.1)


Fig. 1.1

$$x(0) = x_o = X(x_o, 0) \tag{1.1}$$

$$x(t) = x = X(x_o, t). \tag{1.2}$$

Then differentiating (1.2) with respect to time to get the velocity $U(x_o, t)$ of the particle, we have

$$\frac{\partial X(x_o, t)}{\partial t} = U(x_o, t) \tag{1.3}$$

To transform from the Lagrangian to the Eulerian system, solve for $x_o$ in terms of x and t and substitute the function obtained into $U(x_o, t)$ in (1.3). This results in the equation

$$\frac{\partial X}{\partial t} = \frac{dx}{dt} = u(x, t). \tag{1.4}$$

Solving the differential equation (1.4) and choosing $x_o$ as the constant of integration effects the reverse transition from the Eulerian scheme to the Lagrangian.

A brief example serves very well to illustrate these transformations more explicitly. Let the Lagrangian representation be

$$x = X(x_o, t) = ax_o t^2 + x_o, \tag{1.5}$$

then $X(x_o, 0) = x_o$ as indicated in (1.1). Taking the time derivative corresponds to (1.3); thus

$$\frac{\partial X}{\partial t} = 2ax_o t. \tag{1.6}$$

From (1.5), though, $x_o = \frac{x}{at^2 + 1}$; so that (1.6) can be written as

$$\frac{\partial X}{\partial t} = \frac{2axt}{at^2 + 1} = u(x, t).$$

Suppose we are given the Eulerian representation

$$u(x, t) = \frac{2axt}{at^2 + 1},$$

and it is desired to find the Lagrangian representation from this. Equating fluid velocities of the two systems gives

$$\frac{\partial X}{\partial t} = \frac{2axt}{at^2 + 1},$$

and since $X = x$,

$$\frac{\partial X}{\partial t} = \frac{2aXt}{at^2 + 1}. \tag{1.7}$$

Integrating (1.7) yields

$$\log X = \log (at^2 + 1) + \log C$$

hence

$$X = C(at^2 + 1). \tag{1.8}$$

We recall that $X(x_o, 0) = x_o$, hence (1.8) becomes

$$X = x_o(at^2 + 1),$$

which is the same as (1.5).

For motion involving only one space dimension it is generally preferable to write the equations in Lagrangian form, hence in most of the following discussion we will use Lagrange's method. Also, throughout this chapter, the continuity of all the functions and their

derivatives will be assumed.

Suppose the motion we are to be concerned with is that of a compressible fluid in a long pipe with unit cross-sectional area. In order for the motion to be one dimensional, friction between the fluid and the walls of the pipe must be ignored, as must the effect of gravity on the fluid. In addition we shall assume that no other body forces or heat sources are present.

Let us isolate a small, moving cylindrical element of the fluid whose cross-section is equal to that of the pipe. If at $t = 0$ the length of the cylindrical element is $\Delta x_o$, and the left-hand and right-hand ends are at $x_o$ and $x_1$ respectively, then at time $t > 0$, the coordinates of the ends will be $X(x_o, t)$ and $X(x_1, t)$, and the length of the cylinder will be (see Fig. 1.2)

$$\Delta X_o = X(x_1, t) - X(x_o, t)$$

or

$$\Delta X_o = X(x_o + \Delta x_o, t) - X(x_o, t) \quad (1.9)$$

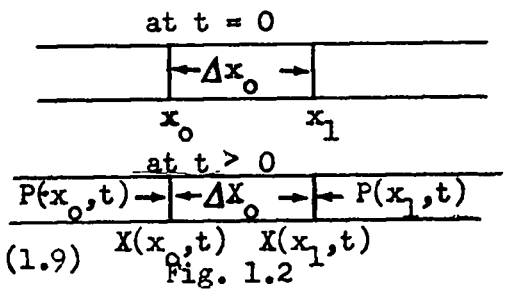

Fig. 1.2

Let the density function be denoted by $R(x, t)$. Take $\bar{R}_o$ to be the initial average density in the cylindrical element and $\bar{R}$ the corresponding quantity at a later time t.*

Now that the preliminary remarks have been made, we can pass on to the actual development of the equations. Conservation of mass is ex-

*The functional dependence of $\bar{R}$ is not noted, but it will be seen to be unimportant.

pressed by equating the mass of the cylinder in its original configuration to its mass at a later time. Hence

$$\frac{\bar{R}_o}{\bar{R}} = \frac{\Delta X_o}{\Delta x_o} \tag{1.10}$$

If we let $x_1$ approach $x_o$ as a limit, $\Delta X_o$ and $\Delta x_o$ approach zero, $\bar{R}_o$ approaches $R(x_o, 0)$, and $\bar{R}$ approaches $R(x_o, t)$. Noting the definition of $\Delta X_o$ given by (1.9), (1.10) becomes

$$\frac{R(x_o, 0)}{R(x_o, t)} = \frac{\partial X(x_o, t)}{\partial x_o} \quad . \tag{1.11}$$

We define the specific volume as

$$V(x_o, t) = \frac{1}{R(x_o, t)} \; ;$$

then (1.11) can be written as

$$V(x_o,t) = \frac{1}{R(x_o,0)} \frac{\partial X(x_o,t)}{\partial x_o}, \quad \underline{\text{The Equation of Continuity}}. \tag{1.12}$$

Conservation of momentum for the small cylindrical element mentioned above requires that the rate of change of momentum of the cylinder equals the total external force applied. The momentum of the element at time t is given by

$$\int_{X(x_o,t)}^{X(x_1,t)} \frac{\partial X(x,t)}{\partial t} \, R(x,t) \, dX. \tag{1.13}$$

Since t is to be held constant during the integration, (1.11) can be used to obtain the differential relationship

$$R(x_o,0)dx = R(x_o,t)dX,$$

and this in turn can be used to transform (1.13) into

$$\int_{x_o}^{x_1} \frac{\partial X(x,t)}{\partial t} \, R(x,0)dx.$$

The rate of change of momentum then is

$$\frac{\partial}{\partial t} \int_{x_o}^{x_1} \frac{\partial X(x,t)}{\partial t} \, R(x,0)dx = \int_{x_o}^{x_1} \frac{\partial^2 X(x,t)}{\partial t^2} \, R(x,0) \, dx;$$

by the first mean-value theorem for definite integrals this becomes

$$R(a,0) \ \Delta x_o \ \frac{\partial^2 X(a,t)}{\partial t^2} ,$$

where $x_o < a < x_1$. The forces present in the gas are assumed to be due only to the pressure; so the total external force applied to the cylindrical element is $P(x_o,t) - P(x_1,t)$, where $P(x,t)$ is the pressure function. Hence the result

$$R(a,0) \, \Delta x_o \, \frac{\partial^2 X(a,t)}{\partial t^2} = P(x_o,t) - P(x_1,t),$$

or

$$R(a,0) \, \frac{\partial^2 X(a,t)}{\partial t^2} = - \frac{P(x_o + \Delta x_o,t) - P(x_o,t)}{\Delta x_o} . \qquad (1.14)$$

In the same limiting process as above, a approaches $x_o$ so that (1.14) becomes

$$\frac{\partial^2 X(x_o,t)}{\partial t^2} = - \frac{1}{R(x_o,0)} \frac{\partial P(x_o,t)}{\partial x_o} , \quad \underline{\text{The Equation of Motion}}. \qquad (1.15)$$

At this point it should be noted that (1.12) and (1.15) are Lagrangian in form, since their development was made from the viewpoint of moving along with the fluid, i.e., a small element of moving fluid was considered. It is of interest to see how these equations can be put into Eulerian form by applying the methods demonstrated in the early part of this chapter.

If both sides of (1.11) are differentiated with respect to time, we have

$$-\frac{1}{R^2}\frac{\partial R}{\partial t} = \frac{1}{R(x_o,0)}\frac{\partial^2 X}{\partial x_o \partial t} = \frac{1}{R(x_o,0)}\frac{\partial U}{\partial x_o}\,.* \qquad (1.16)$$

Using lower case letters to represent Eulerian quantities corresponding to Lagrangian quantities already defined, we have at the point x (See (1.2))

$$R(x_o,t) = r(x,t) \qquad (1.17)$$

$$U(x_o,t) = u(x,t). \qquad (1.18)$$

Since $x = x(t)$, we have from (1.17)

$$-\frac{1}{R^2}\frac{\partial R}{\partial t} = -\frac{1}{r^2}\frac{dr}{dt} = -\frac{1}{r^2}\left[\frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial t}\right],$$

which after application of (1.4) becomes

$$-\frac{1}{R^2}\frac{\partial R}{\partial t} = -\frac{1}{r^2}\left[u\frac{\partial r}{\partial x} + \frac{\partial r}{\partial t}\right]. \qquad (1.19)$$

* From this point on, the argument of the functions will be omitted, i.e., for Lagrangian functions $F = F(x_o,t)$ will be assumed, for Eulerian, $f = f(x,t)$ will be assumed.

Now, from (1.11)

$$\frac{1}{R(x_o,0)}\ \frac{\partial U}{\partial x_o} = \frac{1}{R(x_o,0)}\ \frac{\partial U}{\partial X}\ \frac{\partial X}{\partial x_o} = \frac{1}{R}\ \frac{\partial U}{\partial X}\ ;$$

and from (1.17), (1.18), and (1.2),

$$\frac{1}{R}\ \frac{\partial U}{\partial X} = \frac{1}{r}\ \frac{\partial u}{\partial x}\ . \tag{1.20}$$

It then follows that

$$\frac{1}{R(x_o,0)}\ \frac{\partial U}{\partial x_o} = \frac{1}{r}\ \frac{\partial u}{\partial x}\ . \tag{1.21}$$

Hence from (1.19) and (1.21), (1.16) becomes

$$-\frac{1}{r^2}\left[u\ \frac{\partial r}{\partial x} + \frac{\partial r}{\partial t}\right] = \frac{1}{r}\frac{\partial u}{\partial x}\ ;$$

after multiplying through by $r^2$ and simplifying this is

$$\frac{\partial(ru)}{\partial x} + \frac{\partial r}{\partial t} = 0\ , \tag{1.22}$$

the Eulerian equation of continuity.

The transformation of the equation of motion is somewhat simpler. Application of (1.3), (1.18), (1.2), and (1.4) in that order yields

$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial U}{\partial t} = \frac{du}{dt} = u\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\ . \tag{1.23}$$

By the same reasoning used to obtain (1.21)

$$-\frac{1}{R(x_o,0)}\ \frac{\partial P}{\partial x_o} = -\frac{1}{r}\ \frac{\partial P}{\partial x}\ . \tag{1.24}$$

So (1.23) and (1.24) substituted into (1.15) give us the Eulerian equation of motion

$$-\frac{1}{r}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} \quad . \tag{1.25}$$

Let us carry through the development of the equation of energy conservation in the opposite sense to that above; i.e., derive it in the Eulerian system, then transform it to the Lagrangian. Consider a right cylindrical volume fixed in the pipe whose cross-section equals that of the pipe. The law of conservation of energy can be stated as follows: the rate at which the total energy of the fluid inside this small element of volume changes in time is equal to the sum of the total energy flowing into the element per unit time minus the rate at which the work being done by the volume element changes in time. The total energy inside the cylinder, if we let $x_o$ and $x_1$ be the co-ordinates of the left and right faces of the cylinder, respectively, is

$$\int_{x_o}^{x_1} r(x',t)\left\{1/2\left[u(x',t)\right]^2 + e(x',t)\right\} dx' \ ,$$

where e is the internal energy per unit mass. The rate of change of the total energy is (above terms are abbreviated)

$$\frac{\partial}{\partial t}\int_{x_o}^{x_1} r(u^2/2 + e)\, dx' \ ,$$

which can be rewritten as

$$\int_{x_o}^{x_1} \left[ \left( (1/2)u^2 + e \right) \frac{\partial r}{\partial t} + r \frac{\partial}{\partial t} \left( (1/2)u^2 + e \right) \right] dx',$$

since $x_o$ and $x_1$ are independent of t. The total energy flowing into the element per unit time is given by

$$r(x_o,t)\, u(x_o,t) \left\{ e(x_o,t) + 1/2 \left[ u(x_o,t) \right]^2 \right\}$$
$$- r(x_1,t)\, u(x_1,t) \left\{ e(x_1,t) + 1/2 \left[ u(x_1,t) \right]^2 \right\}$$

and the rate of change of the work done by the element is

$$p(x_1,t)\, u(x_1,t) - p(x_o,t)\, u(x_o,t) .$$

Making use of the above mathematical expressions, the law of conservation of energy can be written as

$$\int_{x_o}^{x_1} \left[ \left( \frac{u^2}{2} + e \right) \frac{\partial r}{\partial t} + r \frac{\partial}{\partial t} \left( \frac{u^2}{2} + e \right) \right] dx'$$
$$= r(x_o,t)\, u(x_o,t) \left\{ e(x_o,t) + 1/2 \left[ u(x_o,t) \right]^2 \right\}$$
$$- r(x_1,t)\, u(x_1,t) \left\{ e(x_1,t) + 1/2 \left[ u(x_1,t) \right]^2 \right\}$$
$$+ p(x_o,t)\, u(x_o,t) - p(x_1,t)\, u(x_1,t). \qquad (1.26)$$

The right-hand side of (1.26) can be rewritten as

$$- \int_{x_o}^{x_1} \frac{\partial}{\partial x'} \left[ ru \left( e + \frac{u^2}{2} \right) + pu \right] dx'$$

so that (1.26) becomes

$$\int_{x_o}^{x_1} \left\{ \left( \frac{u^2}{2} + e \right) \frac{\partial r}{\partial t} + r \frac{\partial}{\partial t} \left( \frac{u^2}{2} + e \right) \right\} dx'$$

$$+ \int_{x_o}^{x_1} \left\{ \frac{\partial}{\partial x'} \left[ ru \left( e + \frac{u^2}{2} \right) \right] + \frac{\partial (pu)}{\partial x'} \right\} dx' = 0 \qquad (1.27)$$

Let

$$F(x) \int_{x_o}^{x} f(x)\, dx = 0 \; ; \qquad (1.28)$$

then, if f(x) is continuous,

$$F'(x) = f(x) = 0 \; . \qquad (1.29)$$

Equation (1.27) is similar to (1.28) in the respect that the limits of integration are independent of the fact that the integral vanishes. Hence by the same reasoning as was used to obtain (1.29), we can set the integrand of (1.27) equal to zero. Carrying out the differentiations, we have

$$\frac{u^2}{2} \frac{\partial r}{\partial t} + e \frac{\partial r}{\partial t} + ru \frac{\partial u}{\partial t} + r \frac{\partial e}{\partial t} + ru \frac{\partial e}{\partial x} + re \frac{\partial u}{\partial x}$$

$$+ ue \frac{\partial r}{\partial x} + \frac{3}{2} ru^2 \frac{\partial u}{\partial x} + \frac{u^3}{2} \frac{\partial r}{\partial x} = -p \frac{\partial u}{\partial x} - u \frac{\partial p}{\partial x} \; . \qquad (1.30)$$

From (1.25)

$$- \frac{\partial p}{\partial x} = ru \frac{\partial u}{\partial x} + r \frac{\partial u}{\partial t} \; . \qquad (1.31)$$

Substituting (1.31) into (1.30) and factoring out the total energy, yields

$$\left(\frac{u^2}{2} + e\right)\left[\frac{\partial r}{\partial t} + r\frac{\partial u}{\partial x} + u\frac{\partial r}{\partial x}\right] + r\frac{\partial e}{\partial t} + ru\frac{\partial e}{\partial x} = -p\frac{\partial u}{\partial x}.$$

From (1.22) it is seen that the term in square brackets vanishes, so that we have

$$r\frac{\partial e}{\partial t} + ru\frac{\partial e}{\partial x} + p\frac{\partial u}{\partial x} = 0. \qquad (1.32)$$

Solving for $\frac{\partial u}{\partial x}$ in (1.22) and substituting $\frac{1}{v}$ for $r$ wherever it occurs gives

$$\frac{\partial u}{\partial x} = r\left[u\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}\right]. \qquad (1.33)$$

Substituting the value obtained for $\frac{\partial u}{\partial x}$ from (1.33) into (1.32) gives after dividing through by $r$, our final result:

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x} + p\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x}\right] = 0. \qquad (1.34)$$

To transform (1.34) into Lagrangian form, we recall from previous discussions that, since $E(x_o,t) = e(x,t)$ and $V(x_o,t) = v(x,t)$ at a certain point,

$$\frac{\partial E(x_o,t)}{\partial t} = \frac{de}{dt} = \frac{\partial e}{\partial x}\frac{dx}{dt} + \frac{\partial e}{\partial t} = u\frac{\partial e}{\partial x} + \frac{\partial e}{\partial t}$$

and

$$\frac{\partial V(x_o,t)}{\partial t} = \frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial t} = u\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}.$$

It then follows, from the above two relationships and (1.34) that

$$\frac{\partial E}{\partial t} = -P\frac{\partial V}{\partial t}, \quad \underline{\text{The Equation of Energy Conservation}}. \qquad (1.35)$$

As was noted on page 10, in addition to equations (1.12), (1.15) and (1.35), an equation of state relating the thermodynamic quantities is necessary to adequately describe the gas motion in mathematical terms. Suppose the fluid we are concerned with is an ideal gas; i.e., a gas in which the thermodynamic functions are related by the equation

$$PV = KT, \tag{1.36}$$

where T is the absolute temperature and K is a constant depending on the molecular weight of the gas. If in addition the gas is polytropic,

$$E = \frac{K}{\gamma-1} T \tag{1.37}$$

where $\gamma$ is a positive constant greater than unity. Substitution of (1.37) into (1.36) gives us

$$P = (\gamma-1)\frac{E}{V}, \quad \text{The Equation of State of the Gas.} \tag{1.38}$$

For later use we note at this point the general thermodynamic relationship

$$dE = -PdV + TdS, \tag{1.39}$$

where S is a thermodynamic function called the entropy. The entropy function expresses the availability of heat energy, and is formally defined by the relation

$$\Delta S = \frac{\Delta Q}{T}, \tag{1.40}$$

where $\Delta S$ is the variation in entropy during a change of state and $\Delta Q$ is the amount of heat absorbed in the same change of state. Comparison of (1.39) and (1.35) verifies a tacit assumption made in the development

of the equations; that the entropy of a specific element of mass remains constant [ 2, pp 46 to 75 ].

Considering E as a function of S and V for the general case, i.e., entropy not necessarily constant, we have

$$dE = \frac{\partial E}{\partial S} dS + \frac{\partial E}{\partial V} dV ; \tag{1.41}$$

which, when compared with (1.39), gives

$$P = - \frac{\partial E}{\partial V}, \quad T = \frac{\partial E}{\partial S} . \tag{1.42}$$

Since $T = \frac{PV}{K}$ from (1.36), we can combine the two equations of (1.42) as

$$K \frac{\partial E}{\partial S} + V \frac{\partial E}{\partial V} = 0. \tag{1.43}$$

Solving (1.43) by the method of separation of variables yields the solution

$$E = \left[ V \exp \left( -\frac{S}{K} \right) \right]^{\alpha} , \tag{1.44}$$

where $\alpha$ is a constant to be determined. From (1.42) then

$$P = - \alpha \left[ V \exp \left( - \frac{S}{K} \right) \right]^{\alpha-1} \exp \left( - \frac{S}{K} \right) \tag{1.45}$$

and

$$T = - \frac{\alpha V}{K} \left[ V \exp \left( - \frac{S}{K} \right) \right]^{\alpha-1} \exp \left( - \frac{S}{K} \right) = - \frac{\alpha}{K} \left[ V \exp \left( - \frac{S}{K} \right) \right]^{\alpha}. \tag{1.46}$$

Substitution of (1.46) and (1.44) into (1.37) gives

$$\left[ V \exp \left( - \frac{S}{K} \right) \right]^{\alpha} = - \frac{\alpha}{\gamma-1} \left[ V \exp \left( - \frac{S}{K} \right) \right]^{\alpha} .$$

It then follows that

$$\alpha = -(\gamma - 1), \text{ and } (1.45) \text{ becomes}$$

$$P = A(S)\, V^{-\gamma}, \quad \underline{\text{The Entropic Equation of State}}, \qquad (1.47)$$

where $A(S) = (\gamma - 1)\left[\exp -\frac{S}{K}\right]^{(\gamma-1)}$. This formula has been developed here for later use.

# CHAPTER III

## SHOCKS

Consider now the propagation of disturbances through ideal gas in a pipe. The same assumptions as were made in Chapter II are made here. These are: (1) that friction and gravity have no effect on the motion of the gas, (2) that there are no other body forces or heat sources present, and (3) that the entropy remains constant for any specific moving element of mass of the gas.

Of particular interest in later discussions will be the sound speed c in the gas [4, pp 200 to 202]. As sound consists of very slight and gradual (spacewise) compressions and expansions, assume in the following discussion of sound speed that the changes of density and velocity, which occur as a result of the sonic disturbance and the space derivatives of these changes, are so small that products of these quantities may be neglected. Hence, when a sonic disturbance moves into a region of ideal gas at rest, even if the term $u \frac{\partial u}{\partial x}$ is neglected in the Eulerian equation of motion (1.25), this equation will still accurately describe the resulting movement of the gas in the region. Then (1.25) becomes simply

$$\frac{\partial u}{\partial t} = - \frac{1}{r} \frac{\partial p}{\partial x} \quad . \tag{2.1}$$

Let the density be represented by

$$r = \bar{r} \left[ 1 + s(x,t) \right] , \tag{2.2}$$

where $\bar{r}$ is the average density of the gas over the region being considered and s(x,t) is a measure of the departures from this average value (since r(x,t) is continuous with continuous derivatives, s(x,t) is also). Putting this expression into the equation of continuity (1.22) yields

$$\bar{r}u \frac{\partial s}{\partial x} + \bar{r} \frac{\partial u}{\partial x} + \bar{r}s \frac{\partial u}{\partial x} + \bar{r} \frac{\partial s}{\partial t} = 0. \qquad (2.3)$$

The first and third terms in (2.3) are second order quantities and may be ignored because of the assumption made above. After dividing through by $\bar{r}$, (2.3) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial s}{\partial t} = 0. \qquad (2.4)$$

Since constant entropy is being assumed, by use of (1.47), p = p(r). Application of the chain rule gives

$$\frac{\partial p}{\partial x} = \frac{dp}{dr} \frac{\partial r}{\partial x} .$$

If (2.2) is differentiated with respect to x and the results are substituted into the above equation, we have

$$\frac{\partial p}{\partial x} = \frac{dp}{dr}\left[\bar{r} \frac{\partial s}{\partial x}\right] = \frac{dp}{dr}\left[r \frac{\partial s}{\partial x} - \bar{r}s \frac{\partial s}{\partial x}\right] \qquad (2.5)$$

(The latter term in (2.5) is the result of another application of (2.2)). However, if second order terms are again neglected, (2.5) simplifies to

$$\frac{\partial p}{\partial x} = r \frac{dp}{dr} \frac{\partial s}{\partial x} . \qquad (2.6)$$

Substitution of (2.6) into (2.1) gives

$$\frac{\partial u}{\partial t} = - \frac{dp}{dr} \frac{\partial s}{\partial x} . \qquad (2.7)$$

From (1.47) and (2.2)

$$\frac{dp}{dr} = \gamma A r^{\gamma-1} = \gamma A \bar{r}^{\gamma-1} [1 + s]^{\gamma-1} . \qquad (2.8)$$

Expanding $[1 + s]^{\gamma-1}$ by the binomial formula, (2.8) becomes

$$\frac{dp}{dr} = \gamma A \bar{r}^{\gamma-1} \left[1 + (\gamma-1)s + \frac{(\gamma-1)(\gamma-2)}{2!} s^2 + \cdots\right]. \qquad (2.9)$$

It has been agreed to neglect second and higher order terms, so (2.9) can be rewritten as

$$\frac{dp}{dr} = \gamma A \bar{r}^{\gamma-1} [1 + (\gamma-1)s] . \qquad (2.10)$$

Substituting the expression obtained for $\frac{dp}{dr}$ from (2.10) into (2.7) yields

$$\frac{\partial u}{\partial t} = - \gamma A \bar{r}^{\gamma-1} \left[1 + (\gamma-1)s\right] \frac{\partial s}{\partial x}$$

$$= - \gamma A \bar{r}^{\gamma-1} \left[\frac{\partial s}{\partial x} + (\gamma-1)s \frac{\partial s}{\partial x}\right] ;$$

it then follows, after discarding $(\gamma-1)s \frac{\partial s}{\partial x}$ and noticing that $\gamma A \bar{r}^{\gamma-1} = \frac{\gamma p(\bar{r})}{\bar{r}}$ , that

$$\frac{\partial u}{\partial t} = - \frac{\gamma p(\bar{r})}{\bar{r}} \frac{\partial s}{\partial x} = - \left[\frac{dp}{dr}\right]_{r=\bar{r}} \frac{\partial s}{\partial x} . \qquad (2.11)$$

Now, differentiating (2.4) with respect to t, it is seen that

$$\frac{\partial^2 u}{\partial x \partial t} = - \frac{\partial^2 s}{\partial t^2} ; \qquad (2.12)$$

and differentiating (2.11) with respect to x gives, since $\left[\frac{dp}{dr}\right]_{r=\bar{r}}$ is independent of x and t,

$$\frac{\partial^2 u}{\partial t \partial x} = -\left[\frac{dp}{dr}\right]_{r=\bar{r}} \frac{\partial^2 s}{\partial x^2} \quad . \tag{2.13}$$

Subtracting (2.12) from (2.13) yields the relationship

$$\frac{\partial^2 s}{\partial t^2} - \left[\frac{dp}{dr}\right]_{r=\bar{r}} \frac{\partial^2 s}{\partial x^2} = 0 \, , \tag{2.14}$$

the differential equation of a wave motion propagated with velocity $\left\{\left[\frac{dp}{dr}\right]_{r=\bar{r}}\right\}^{1/2}$. Since $s(x,t)$ is defined as the variance from the mean density caused by a sonic disturbance, it is seen from (2.14) that a sonic disturbance is propagated in wave form. The velocity of this disturbance is given by

$$c = \left\{\left[\frac{dp}{dr}\right]_{r=\bar{r}}\right\}^{1/2} \left[\frac{\gamma p(\bar{r})}{\bar{r}}\right]^{1/2},$$

and is traditionally known as the sound speed. It is convenient, then, to define the <u>local sound speed</u>

$$c(x,t) = \left[\frac{dp}{dr}\right]^{1/2} = \left[\frac{\gamma p(x,t)}{r(x,t)}\right]^{1/2} \quad . \tag{2.15}$$

A small disturbance moving in a neighborhood of x at time t will travel at approximately the velocity c.

The effect of a large disturbance that causes the density and velocity to change in such a manner that

$$r = r(u)^* \tag{2.16}$$

will now be studied.

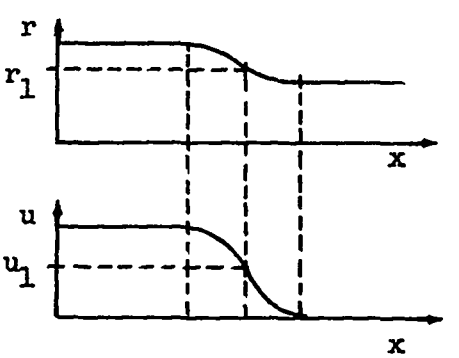


Fig. 2.1

*The fact that such disturbances do exist is seen from the discussions in reference 1 on pages 45 to 48 and 80 to 92.

Figure 2.1 shows, at a specific time t, the shape of the velocity and density curves resulting from such a disturbance. Since the disturbance is large, the resulting changes are considerable and occur rapidly; therefore, the assumption that second order terms are negligible, made in the development of the sound speed, is not valid here. So, in order to describe the propagation of large disturbances, it is necessary to treat the equations (1.22) and (1.25) in a somewhat different manner.

Since from (2.16) $r$ does not depend on $t$ explicitly, we can rewrite (1.22), by use of the chain rule, as

$$\frac{dr}{du}\left[\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x}\right] = -\,r\,\frac{\partial u}{\partial x} \quad . \tag{2.17}$$

Also, after looking at (1.47), one notices that $\frac{\partial p}{\partial x} = \frac{dp}{dr}\,\frac{dr}{du}\,\frac{\partial u}{\partial x}$ .

Hence multiplying both sides of (1.25) by $\frac{dr}{du}$ yields

$$\frac{dr}{du}\left[\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x}\right] = -\,\frac{1}{r}\left[\frac{dr}{du}\right]^2 \frac{dp}{dr}\,\frac{\partial u}{\partial x} \quad . \tag{2.18}$$

Subtracting (2.17) from (2.18) gives

$$r\,\frac{\partial u}{\partial x} - \frac{1}{r}\left[\frac{dr}{du}\right]^2 \frac{dp}{dr}\,\frac{\partial u}{\partial x} = 0 \;;$$

then, after multiplying through by $r\left[\frac{dp}{dr}\,\frac{\partial u}{\partial x}\right]^{-1}$ and noticing from (2.15) that $\frac{dp}{dr} = c^2$,

$$\left[\frac{dr}{du}\right]^2 - \frac{r^2}{c^2} = 0 \quad . \tag{2.19}$$

Equation (2.19) is a quadratic equation in $\frac{dr}{du}$ with solutions

$$\frac{dr}{du} = \pm\,\frac{r}{c} \quad . \tag{2.20}$$

Substituting (2.20) into (2.17) and multiplying through by $\pm \frac{c}{r}$ yields

$$\frac{\partial u}{\partial t} + (u \pm c) \frac{\partial u}{\partial x} = 0. \tag{2.21}$$

Equation (2.18) can be rewritten as

$$\frac{\partial r}{\partial t} + u \frac{\partial r}{\partial x} = - \frac{c^2}{r} \frac{\partial r}{\partial x} \frac{dr}{du} ,$$

which becomes, after substitution of $\pm \frac{r}{c}$ for $\frac{dr}{du}$ by use of (2.20),

$$\frac{\partial r}{\partial t} + (u \pm c) \frac{\partial r}{\partial x} = 0. \tag{2.22}$$

Consider the case when the plus sign is chosen in (2.21) and (2.22). It can be shown by direct differentiation, after noting from (2.15) and (2.16) that $c = c(u)$, that

$$u = F[x - (u + c)t] \tag{2.23}$$

and

$$r = G[x - (u + c)t], \tag{2.24}$$

where $F$ and $G$ are arbitrary functions, are solutions of (2.21) and (2.22) respectively. These solutions resemble wave motion to the right, in that it is necessary for an observer to move at a constant speed if he desires to follow a point at which $u$ = constant. The solutions (2.23) and (2.24) actually describe nonlinear wave motion, which, as will be seen below, exhibits quite different properties from the linear wave motion mentioned earlier (see equation (2.14)).

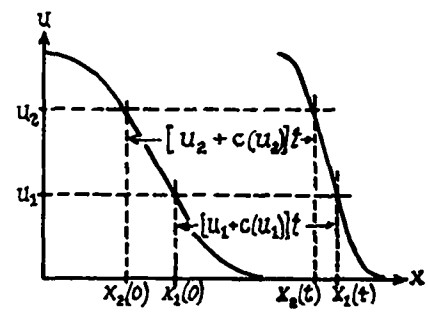

Fig. 2.2

Suppose one follows two points at which u = constant on the curve representing the velocity resulting from the disturbance propagation (see Figure 2.2). Let the point where $u = u_1$ be denoted by $x_1(t)$; then from (2.23),

$$u_1 = F\left\{x_1(t) - \left[u_1 + c(u_1)\right] t\right\}.$$

In other words, $x_1(t)$ moves with constant speed $u_1 + c(u_1)$, as was stated above. Similarly, for some constant velocity $u_2 > u_1$,

$$u_2 = F\left\{x_2(t) - \left[u_2 + c(u_2)\right]t\right\};$$

so that $x_2(t)$ moves at velocity $u_2 + c(u_2)$.

Consider now the velocity function at some time t. It is important to investigate the possibility that the velocities $u_1 + c(u_1)$ and $u_2 + c(u_2)$ of $x_1(t)$, and $x_2(t)$, respectively, may not be equal. If it can be shown that

$$\frac{dc}{du} > 0, \tag{2.25}$$

then it will follow easily that $u_1 + c(u_1)$ and $u_2 + c(u_2)$ are not equal. Consider the derivative $\frac{dc}{dr}$. It is seen by use of (2.15) that

$$\frac{dc}{dr} = \frac{d}{dr}\left(A\gamma r^{\gamma-1}\right)^{1/2} = 1/2\left(A\gamma r^{\gamma-1}\right)^{-1/2} A\gamma(\gamma-1)r^{\gamma-2}$$

$$= \frac{c}{2r}\left[\gamma-1\right] > 0. \tag{2.26}$$

Since the plus sign was chosen in (2.21) and (2.22), equation (2.20) yields the fact that

$$\frac{dr}{du} > 0. \tag{2.27}$$

Hence from (2.26) and (2.27),

$$\frac{dc}{du} = \frac{dc}{dr}\frac{dr}{du} > 0, \qquad (2.28)$$

as was to be shown. Relation (2.28) further shows that sound speed increases with velocity; hence, since $u_2 > u_1$ ,

$$c(u_2) > c(u_1)$$

and

$$u_2 + c(u_2) > u_1 + c(u_1).$$

The wave form will change, due to its non-linearity, in the manner indicated in Figure 2.2. As the disturbance proceeds, at time $t = t_s$, a portion of the wave front will become vertical. From this point on the solution discussed above breaks down, as it shall no longer be possible to obtain a unique velocity at a given point (x, t). However, it is possible to continue using the equations developed in Chapter II to describe the fluid motion by assuming that at time $t = t_s$ the velocity function takes the form indicated in Figure 2.3. This is a finite or ordinary discontinuity moving at a velocity $U_s$ across which the velocity function suffers a sudden jump from $u_1$ to $u_2$. To continue, after the assumption mentioned above is made, certain boundary conditions can be imposed to connect the values of the velocity, density, pressure, and energy functions across the discontinuity; then, the equations

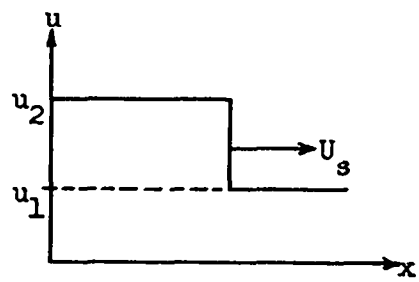


Fig. 2.3

already developed can be used to describe the gas motion in continuous regions of flow (the discontinuity discussed here will often be referred to as a shock in later discussions). The plausibility of the above assumption has been demonstrated by the proof that the wave form produced by a large disturbance steepens as it moves through the gas. It can be shown mathematically [1, pp 108 to 109] that the flow must become discontinuous. This is accomplished by demonstrating that any continuous flow adjacent to a constant state is one satisfying the relationship (2.16), then combining this with the results obtained on page 33. The proof of this, however, is beyond the scope of this paper and hence will be omitted. The interesting historical developments in this work should also be referred to at this point [1, pp 118 to 119] .

The boundary, or "jump" conditions mentioned above, which will connect the values of the velocity and thermodynamic functions across shocks, are derived from the three basic principles of conservation of mass, momentum, and energy. Consider a moving cylindrical element like the one used in Chapter II to develop equations (1.12) and (1.15). If $X(x_o,t)$ and $X(x_1,t)$ denote the positions of the moving particles that form the ends of the cylinder, using (1.12) these co-ordinates can be rewritten as

$$X(x_o,t) = x^o(t) = x^o$$

$$X(x_1,t) = x^1(t) = x^1.$$

The superscripts, although an apparent change of notation, are merely a means to indicate that the Eulerian co-ordinates $x^o$ and $x^1$ are different.

Assuming the flow to be continuous at the ends of the column, the three basic principles are expressed as follows [1, p 122]:

$$\frac{d}{dt}\int_{x^o(t)}^{x^1(t)} r\,dx = 0 \quad \text{(conservation of mass);} \tag{2.29}$$

$$\frac{d}{dt}\int_{x^o(t)}^{x^1(t)} ru\,dx = p(x^o,t) - p(x^1,t) \quad \text{(conservation of momentum);} \tag{2.30}$$

$$\frac{d}{dt}\int_{x^o(t)}^{x^1(t)} r\left(\frac{u^2}{2} + e\right) dx = p(x^o,t)\,u(x^o,t) - p(x^1,t)\,u(x^1,t)$$

$$\text{(conservation of Energy)}^{*} \tag{2.31}$$

Suppose that a shock exists at some point $x = z(t)$ such that $x^o < x < x^1$ and whose velocity is $U_s = \frac{dz}{dt}$. Mathematically this shock will be a discontinuity in the pressure, density, energy, and velocity functions and will be of the type shown in Figure 2.3, owing to earlier assumptions. The integrals in (2.29), (2.30), and (2.31) all have the form

$$I = \int_{x^o(t)}^{x^1(t)} g(x,t)\,dx,$$

where g is discontinuous at $x = z$ in the manner of Figure 2.3. Differentiation and application of Liebnitz' rule lead to

* For continuous flow within the interval $(x^o,x^1)$, the Eulerian equations of gas flow can be obtained directly from (2.29), (2.30), and (2.31).

$$\frac{dI}{dt} = \frac{d}{dt}\int_{x^o(t)}^{z(t)} g(x,t)\,dx + \frac{d}{dt}\int_{z(t)}^{x^1(t)} g(x,t)\,dx$$

$$= \int_{x^o(t)}^{x^1(t)} \frac{\partial g(x,t)}{\partial t}\,dx + g_o\frac{dz}{dt} - g(x^o,t)\left[\frac{dx}{dt}\right]_{x=x^o} + g(x^1,t)\left[\frac{dx}{dt}\right]_{x=x^1} - g_1\frac{dz}{dt}, \tag{2.32}$$

where the quantities $g_1$ and $g_o$ are the limits of $g(x,t)$ as $x$ approaches $z$ from the sides $x < z$ and $x > z$ respectively. Now, performing the limiting process by letting $x^o$ and $x^1$ approach $z$, the first integral on the right-hand side of (2.32) approaches zero, $g(x^1,t) \to g_o$, $g(x^o,t) \to g_1$; and, because of the nature of the discontinuity, $\left[\frac{dx}{dt}\right]_{x=x^o} \to u_1 \neq U_s$ and $\left[\frac{dx}{dt}\right]_{x=x^1} \to u_o \neq U_s$. Hence,

$$\lim_{\substack{x^1 \to z \\ x^o \to z}} \frac{dI}{dt} = g_1 u'_1 - g_o u'_o \tag{2.33}$$

where

$$u'_i = u_i - U_s \quad (i = 0,1). \tag{2.34}$$

The subscripts 0 and 1 denote the value of the quantities directly in front of and behind the shock, respectively. Thus the jump conditions may be written in the order of conservation of mass, momentum and energy, as follows (because of the great number of formulas in the following discussion, the important ones will be underlined):

$$r_1 u'_1 = r_o u'_o \quad \text{(conservation of mass)}; \tag{2.35}$$

$$(r_1 u_1)\, u'_1 - (r_o u_o) u'_o = p_o - p_1 \quad \text{(conservation of momentum)}; \tag{2.36}$$

or, after substituting for $u_i$ the value $u'_i + U_s (i = 0,1)$, obtained from (2.34), and noticing that, from (2.35), $r_1 u'_1 U_s = r_o u'_o U_s$,

$$r_1 (u'_1)^2 + p_1 = r_o (u'_o)^2 + p_o; \tag{2.37}$$

and

$$r_1 u'_1 (u_1^2/2 + e_1) - r_o u'_o (u_o^2/2 + e_o) = p_o u_o - p_1 u_1 \quad \text{(conservation of energy)}. \tag{2.38}$$

Substituting $u'_i + U_s$ for $u_i (i = 0,1)$ in (2.38), and rearranging gives

$$r_1 u'_1 \left[(1/2)(u'_1)^2 + u'_1 U_s + (1/2)\, U_s^2 + e_1\right] + p_1 u'_1 + p_1 U_s$$

$$= \left[(1/2)(u'_o)^2 + u'_o U_s + (1/2)\, U_s^2 + e_o\right] r_o u'_o + p_o u'_o + p_o U_s . \tag{2.39}$$

It is seen from multiplying (2.35) through by $(1/2)\, U_s^2$ and (2.37) through by $U_s$ and the fact that $r_i v_i = 1 (i = 0,1)$, that (2.39) is the same as

$$r_1 u'_1 \left[(1/2)(u'_1)^2 + e_1 + p_1 v_1\right] = r_o u'_o \left[(1/2)\,(u'_o)^2 + e_o + p_o v_o\right] . \tag{2.40}$$

From (2.37) and (2.35)

$$p_1 - p_o = r_o u'_o (u'_o - u'_1).$$

Multiplying both sides by $v_o + v_1$ and again applying (2.35),

$$(p_1 - p_o)(v_o + v_1) = (r_o u'_o v_o + r_1 u'_1 v_1)(u'_o - u'_1) = (u'_o)^2 - (u'_1)^2. \tag{2.41}$$

Dividing (2.40) through by $r_o u'_o$, taking note of (2.35), and eliminating $u'_o$ and $u'_1$ from the result and (2.41) gives

$$(1/2)\ (v_o - v_1)(p_o + p_1) = e_1 - e_o\ . \tag{2.42}$$

Relation (2.42) was first introduced by Hugoniot, hence bears his name. It is noteworthy since it refers only to thermodynamical quantities.

Substitution of (1.38) into (2.42) yields

$$(1/2)(v_o - v_1)(p_o + p_1) = \frac{1}{\gamma-1}(p_1 v_1 - p_o v_o)\ ;$$

then, when pressures are factored out and terms are combined, we have

$$(1/2)p_o\left[\frac{1}{m}\, v_o - v_1\right] = (1/2)p_1\left[\frac{1}{m}\, v_1 - v_o\right],$$

where $m = \frac{\gamma-1}{\gamma+1}$ .

From this we easily obtain the Hugoniot relation for polytropic gases,

$$p_o\ (v_o - m v_1) = p_1\ (v_1 - m v_o). \tag{2.43}$$

Using (2.43) it is easy to obtain the formulas

$$\frac{p_1}{p_o} = \frac{v_o - m v_1}{v_1 - m v_o} \tag{2.44}$$

and

$$\frac{v_o}{v_1} = \frac{p_1 + m p_o}{p_o + m p_1}\ . \tag{2.45}$$

Since $r = \frac{1}{v}$, it can quickly be seen from (2.37) and (2.35) that the

relation

$$(r_o u'_o)^2 = -\frac{p_o - p_1}{v_o - v_1} \tag{2.46}$$

holds. From (2.46)

$$r_o(u'_o)^2\left[1 - \frac{v_1}{v_o}\right] = p_1 - p_o \ ;$$

which, by considering (2.45) and multiplying through by $p_1 + mp_o$, becomes

$$r_o(u'_o)^2[p_1 + mp_o - p_o - mp_1] = (p_1 - p_o)(p_1 + mp_o). \tag{2.47}$$

The term in square brackets in (2.47) is $(p_1 - p_o)(1 - m)$, so dividing both sides of (2.47) by $p_1 - p_o$ gives

$$r_o(u'_o)^2 (1-m) = (r_o u'_o)^2 (1-m)\, v_o = p_1 + mp_o$$

or

$$(r_o u'_o)^2 = \frac{p_1 + mp_o}{(1-m)\, v_o} \ . \tag{2.48}$$

By the same reasoning,

$$(r_1 u'_1)^2 = \frac{p_o + mp_1}{(1-m)\, v_1} \ . \tag{2.49}$$

Also, a formula that will be needed later on is logically developed at this point: from (2.35) it is possible to interchange $r_o u'_o$ and $r_1 u'_1$ in (2.37) to obtain

$$u'_o u'_1 = \frac{p_o - p_1}{r_o - r_1} \ . \tag{2.50}$$

---

The Mach Number $M_i$ is defined as

$$M_i = \frac{|u'_i|}{c_i} \quad (i = 0,1). \tag{2.51}$$

It is possible to develop a relationship between $M_o$ and the pressure ratio $\frac{p_1}{p_o}$. From $r_o c_o^2 = \gamma p_o$ and (2.48),

$$p_1 + mp_o = (1-m)\, r_o (u'_o)^2 = \gamma(1-m)\, p_o M_o^2$$

or

$$\frac{p_1}{p_o} = (1 + m)M_o^2 - m \,. \tag{2.52}$$

From (2.50) the reasoning is similar to obtain

$$\frac{p_o}{p_1} = (1 + m)M_1^2 - m \,. \tag{2.53}$$

Looking at (1.38) and (2.35), one notices that (2.40) can be rewritten as

$$(1/2)(u'_1)^2 + \frac{\gamma}{\gamma-1} p_1 v_1 = (1/2)(u'_o)^2 + \frac{\gamma}{\gamma-1} p_o v_o;$$

which, after multiplying both sides by 2m and taking note of (2.15), becomes

$$m(u'_1)^2 + (1 - m)c_1^2 = m(u'_o)^2 + (1 - m)c_o^2. \tag{2.54}$$

Both sides of (2.54) can be set equal to some number $N$ to be determined later. Hence

$$m(u'_1)^2 + (1 - m)c_1^2 = N, \quad m(u'_o)^2 + (1 - m)c_o^2 = N. \tag{2.55}$$

Now, from (2.15) and (2.55),

$$m\left[r_1(u'_1)^2 + p_1\right] + p_1 = r_1\left[m(u'_1)^2 + (1 - m)\, c_1^2\right] = r_1 N \quad (2.56)$$

and

$$m\left[r_o(u'_o)^2 + p_o\right] + p_o = r_o\left[m(u'_o)^2 + (1 - m)\, c_o^2\right] = r_o N \quad (2.57)$$

Subtracting (2.57) from (2.56) and noticing from (2.37) that the terms in square brackets are equal, it is seen that

$$p_1 - p_o = N(r_1 - r_o);$$

thus implying, from (2.50), that $N = u'_o u'_1$. Hence from (2.55)

$$\underline{m(u'_1)^2 + (1 - m)c_1^2 = m(u'_o)^2 + (1 - m)c_o^2 = u'_o u'_1.} \quad (2.58)$$

This relation is due to Prandtl. Substituting $u_i - U_s$ for $u'_i$ in (2.58) gives

$$(u_1 - U_s)(u_o - U_s) = m\,(u_o - U_s)^2 + (1 - m)c_o^2$$

$$= m\,(u_1 - U_s)^2 + (1 - m)c_1^2. \quad (2.59)$$

Multiplying (2.59) out, using the extreme left- and right-hand terms, gives

$$u_o u_1 - u_o U_s - u_1 U_s + U_s^2 - m u_1^2 + 2m u_1 U_s - m U_s^2 = (1 - m)c_1^2.$$

Rewriting the term $-u_1 U_s$ as $u_1 U_s - 2u_1 U_s$ and adding and subtracting $u_1^2$ yields

$$-m U_s^2 + 2m u_1 U_s - m u_1^2 + U_s^2 - 2u_1 U_s + u_1^2 - u_o U_s + u_1 U_s + u_o u_1 - u_1^2$$

$$= (1 - m)c_1^2.$$

This can be grouped as

$$(1 - m)(U_s - u_1)^2 - (u_o - u_1)(U_s - u_1) - (1 - m)c_1^2 = 0$$

or

$$\underline{(U_s - u_1)^2 - \frac{1}{1 - m}(u_o - u_1)(U_s - u_1) - c_1^2 = 0} \qquad (2.60)$$

In a similar manner, the relationship

$$\underline{(U_s - u_o)^2 - \frac{1}{1 - m}(u_1 - u_o)(U_s - u_o) - c_o^2 = 0} \qquad (2.61)$$

can be obtained.

A problem of some importance will next be discussed, namely, the reflection of a shock from a rigid wall. This discussion will be concerned with the following mathematical problem: to find functions u, p, r, and e which

1. Are solutions to the differential equations (1.22), (1.25), and (1.34).
2. Satisfy the equation of state.
3. Satisfy boundary and initial conditions (to be described below).
4. Satisfy the jump conditions across the shock.

Consider a region of flow $0 \leq x \leq x_L$ through which a shock moves (see Figure 2.4). Take as the initial conditions the state of the gas in the region ahead of the shock. If the gas is at rest in this region, these conditions are (in this case, $p_o$, $r_o$, and $c_o$ are all constant)

$$u = u_o = 0, \; p = p_o, \; r = r_o, \; e = e_o \qquad (2.62)$$

at all points x such that $z < x \leq x_L$ (z is the position of the shock at time t, as on page 35). The values $p_o$, $e_o$, and $r_o$ are related to each

other according to the equation of state (1.38).

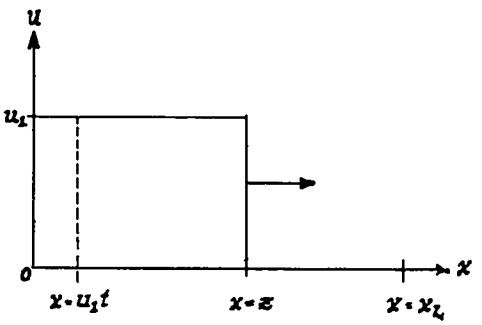

Fig. 2.4

For the left-hand boundary condition, let it be required that the gas particles, which at time $t = 0$ are at the location $x = 0$, move toward the right with velocity $u_1$ (this is actually a Lagrangian boundary condition, so it is somewhat difficult to express in Eulerian form). This can be expressed mathematically as:

$$\text{at } x = u_1 t, \quad u = u_1 . \tag{2.63}$$

The right-hand boundary condition is:

$$\text{at } x = x_L , \quad u = u_o . \tag{2.64}$$

Let the time at which the shock reaches the point $x = x_L$ be $t = t_L$. Then, for $t < t_L$, it is easily verified that the functions designated below by (2.65) through (2.69) give a solution to the problem with initial conditions (2.62) and boundary conditions (2.63) and (2.64). Furthermore, this solution satisfies all the requirements on page 42. The solution depicted above is

$$u = \begin{cases} u_o \text{ for } x \text{ such that } x_L \geq x > z \\ u_1 \text{ for } x \text{ such that } u_1 t \leq x < z \end{cases} * \tag{2.65}$$

also, trivially,

* The interval $(u_1 t, z)$ will increase in width with time, as will be shown later.

$$\lim_{x \to z^+} u = u_o, \quad \lim_{x \to z^-} u = u_1. \tag{2.66}$$

The value $u_1$ is obtained from the left-hand boundary condition. Equations (2.65) and (2.66) completely determine the velocity function. To continue with the writing of the particular solution mentioned above,

$$p = \begin{cases} p_o \text{ for } x \text{ such that } x_L \geq x > z \\ p_1 \text{ for } x \text{ such that } u_1 t \leq x < z \end{cases},$$

$$\lim_{x \to z^+} p = p_o, \quad \lim_{x \to z^-} p = p_1; \tag{2.67}$$

$$r = \begin{cases} r_o \text{ for } x \text{ such that } x_L \geq x > z \\ r_1 \text{ for } x \text{ such that } u_1 t \leq x < z \end{cases},$$

$$\lim_{x \to z^+} r = r_o, \quad \lim_{x \to z^-} r = r_1; \tag{2.68}$$

$$e = \begin{cases} e_o \text{ for } x \text{ such that } x_L \geq x > z \\ e_1 \text{ for } x \text{ such that } u_1 t \leq x < z \end{cases},$$

$$\lim_{x \to z^+} e = e_o, \quad \lim_{x \to z^-} e = e_1; \tag{2.69}$$

where $p_1$, $r_1$, and $e_1$ are constants whose values we will now consider. The velocity function is constant behind the shock, so that the shock velocity $U_+$ can be found by (2.61). Once the shock velocity is determined, $r_1$, $p_1$, and $e_1$, in that order, can be quickly found by use of equations (2.35), (2.36), and either (2.38) or the equation of state. An example using actual numerical values for all of the quantities involved will be done in Chapter V. A slightly different method of finding the pressure function (which often is the only quantity of interest

in problems such as this) utilizing results on page 40 will be employed shortly.

At time $t = t_L$ the situation changes somewhat, although the mathematical problem remains the same. Because of the boundary condition specifying that the velocity at $x = x_L$ must be zero, the shock can no longer move to the right. Otherwise, conservation of mass would be immediately violated (see equation (2.35)). Hence, at time $t = t_L$, the direction of the shock is abruptly reversed. At this time the initial conditions are represented as the state of the gas which was for $0 < t < t_L$ behind the incident shock, but which is for $t > t_L$ in front of the reflected shock. Then, using the same boundary conditions (2.63) and (2.64), the velocity $U_-$ of the reflected shock and constant values of u, p, r, and e behind this shock can be obtained by exactly the same method as above. This mathematical problem arises from the physical case where a piston moves with constant velocity $u_1$ into a tube containing ideal gas which is initially at rest, the tube being plugged at some point ahead of the piston. Hence, it is possible to represent mathematically the flow of gas resulting from this physical situation by piecewise constant functions which are solutions of the differential equations and which satisfy all the conditions.

The situation can be further clarified by a diagram in the (x,t)-plane (Figure 2.5). State (0) is a zone of gas at rest characterized by the quantities $u_o = 0$, $p_o$, $r_o$, $e_o$, and $c_o$. Figure 2.5 shows also a state (1) connected through a shock with the zone of rest (0)

$$M_+ M_- = -1 ; \tag{2.71}$$

so it is seen that $M_+ < 0$. This is of interest, since it gives the relationship $u_1 < U_+$.

Moreover, the pressure relations following from (2.53) are

$$\frac{p_o}{p_1} = (1 + m)M_+^2 - m, \quad \frac{p_2}{p_1} = (1 + m)M_-^2 - m; \tag{2.72}$$

multiplying the two equations of (2.72) together and taking (2.71) into consideration, the relationship

$$\frac{p_o p_2}{p_1^2} = 1 + 2m + m^2 - (m + m^2)(M_+^2 + M_-^2) + m^2 \tag{2.73}$$

is obtained.

Since from (2.72)

$$M_+^2 = \frac{\frac{p_o}{p_1} + m}{1 + m} \quad \text{and} \quad M_-^2 = \frac{\frac{p_2}{p_1} + m}{1 + m} , \tag{2.74}$$

substitution of these values into (2.73) gives

$$\frac{p_o p_2}{p_1^2} = 1 + 2m + m^2 - m\left[\frac{p_o}{p_1} + m + \frac{p_2}{p_1} + m\right] + m^2$$

$$= 1 + 2m - \frac{p_o}{p_1} m - \frac{p_2}{p_1} m;$$

so that

$$\frac{p_2}{p_1} = \frac{p_1}{p_o}(1 + 2m) - m - \frac{p_1 p_2}{p_o p_1} m$$

or

$$\frac{p_2}{p_1} = \frac{\frac{p_1}{p_o}(1 + 2m) - m}{1 + \frac{p_1}{p_o} m} , \qquad (2.75)$$

the formula for the reflected pressure ratio.

This completes our discussion of shocks. Later on, numerical results will be compared with the most important formulas obtained in this chapter.

# CHAPTER IV

## THE VON NEUMANN-RICHTMYER METHOD OF HANDLING SHOCKS

In Chapter III it was shown that a large disturbance, moving into ideal gas in a tube, quickly develops into a shock when the equations derived in Chapter II are used to describe the gas motion. As was seen, a shock manifests itself as a surface across which the fluid velocity, pressure, density, and energy functions have discontinuities; therefore the partial differential equations governing the motion require boundary conditions connecting the values of these quantities on the two sides of the surface. The necessary boundary conditions are supplied by the jump conditions discussed in the latter part of Chapter III. In certain cases, however, such as solving the equations using stepwise numerical procedures, imposition of these boundary conditions is difficult. The Von Neumann-Richtmyer method [3] of handling shocks removes this difficulty by eliminating the discontinuities in the functions. When this method is used, the shocks are seen to be smeared out, so that the mathematical surface of discontinuity is replaced by a thin layer in which pressure, velocity, density, and energy vary rapidly but continuously. This is achieved by introducing an artificial dissipative term into the equations in a manner to be described later. In this chapter the Von Neumann-Richtmyer method will be discussed in detail.

Suppose we substitute for the pressure function $P(x_o,t)$, at every place where it occurs in the differential equations describing fluid motion, the term $P(x_o,t) + Q(x_o,t)$, where $Q(x_o,t)$ is the dissipative term to be defined below. Then equations (1.12), (1.15), and (1.35) can be rewritten as (the Lagrangian form of the equations will be used from here on, so it will not be confusing to eliminate the use of subscripts to indicate Lagrangian co-ordinates; hence, $x_o$ will become simply x)

$$V = \frac{1}{R(x,0)} \frac{\partial X}{\partial x}, \tag{3.1}$$

$$\frac{\partial^2 X}{\partial t^2} = - \frac{1}{R(x,0)} \left[ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x} \right], \tag{3.2}$$

$$\frac{\partial E}{\partial t} = - P \frac{\partial V}{\partial t} - Q \frac{\partial V}{\partial t}, \tag{3.3}$$

respectively.

The dissipation is introduced for mathematical reasons.* Therefore, Q may be taken as any convenient function of P, V, etc. and their derivatives, provided the following requirements are met:

1. The equations (3.1), (3.2), and (3.3) must possess solutions without discontinuities.
2. The effect of the terms containing Q in (3.2) and (3.3) must be negligible outside the shock layer.

*The introduction of Q is reasonable from the physical viewpoint since it is related to the viscosity of the gas. This tends to smear out the shock because forces (those of friction) additional to those due to pressure variations are introduced [1, pp 116 to 117] .

3. The Hugoniot relation (2.42) must hold across the smeared shock.

4. The width of the smeared shock must be small.

It will be shown that the expression

$$Q = - \frac{k^2[R(x,0)]^2}{V} \frac{\partial V}{\partial t} \left| \frac{\partial V}{\partial t} \right| , \tag{3.4}$$

where the constant k determines the approximate shock width, meets the above requirements for the specific problem first considered in Chapter III on page 42. Only the case of the incident shock will be studied since it was seen that the reflected shock behaved in exactly the same manner.

For convenience, the boundary conditions of the problem being considered are restated here (in Lagrangian form):

$$U(0,t) = U_1, \; U(x_L,t) = U_o = 0. \tag{3.5}$$

For the case where the dissipation is included, additional boundary conditions must be imposed. These are:

$$Q(0,t) = 0, \; Q(x_L,t) = 0. \tag{3.6}$$

Looking at the solutions (2.65)--(2.69), one notices that the hydrodynamic functions depend on x and t only through the combination

$$w = x - U_s t, \tag{3.7}$$

where $U_s$ is a constant, for it is a solution to equation (2.61) which has constant coefficients. In order for the solutions to the problem when the dissipative term is included to agree with the solutions

(2.65)--(2.69) outside the shock layer, we will require that P, V, U, etc. also depend only on w.

It is convenient to define

$$A = U_s R(x,0); \tag{3.8}$$

whereupon equation (3.1) becomes, after differentiating both sides with respect to t,

$$-A\frac{dV}{dw} = \frac{dU}{dw}, \tag{3.9}$$

and (3.2) and (3.3) become respectively

$$A\frac{dU}{dw} = \frac{d}{dw}(P + Q) \tag{3.10}$$

and

$$\frac{dE}{dw} + (P + Q)\frac{dV}{dw} = 0. \tag{3.11}$$

Then, (3.9) and (3.10) give

$$-A^2\frac{dV}{dw} = \frac{d}{dw}(P + Q). \tag{3.12}$$

Multiplying (3.12) through by V and adding to (3.11) gives

$$\frac{dE}{dw} + \frac{d}{dw}[(P + Q)V] + A^2V\frac{dV}{dw} = 0. \tag{3.13}$$

It is easy to integrate (3.9), (3.12), and (3.13) to obtain

$$AV + U = C_0, \tag{3.14}$$

$$A^2V + P + Q = C_1, \tag{3.15}$$

$$E + (P + Q)V + \frac{1}{2}A^2V^2 = C_2. \tag{3.16}$$

Suppose the values of P, V, and E at the boundary points are

given by the following relationships:

$$P(0,t) = P_1,\ V(0,t) = V_1,\ E(0,t) = E_1;$$
$$P(x_L,t) = P_o,\ V(x_L,t) = V_o,\ E(x_L,t) = E_o. \tag{3.17}$$

The quantities $P_o$, $V_o$, and $E_o$ are related to each other according to the equation of state, as are $P_1$, $V_1$, and $E_1$. From (3.15) then, by use of the boundary conditions (3.6),

$$A^2(V_o - V_1) = P_1 - P_o. \tag{3.18}$$

Similarly, from (3.16),

$$\begin{aligned} E_1 - E_o + P_1V_1 - P_oV_o &= (\tfrac{1}{2})A^2(V_o^2 - V_1^2) \\ &= (\tfrac{1}{2})A^2(V_o - V_1)(V_o + V_1). \end{aligned} \tag{3.19}$$

Substituting (3.18) into (3.19) and multiplying out, we have

$$E_1 - E_o = P_oV_o - P_1V_1 + (\tfrac{1}{2})P_1V_o + (\tfrac{1}{2})P_1V_o - (\tfrac{1}{2})P_oV_o - (\tfrac{1}{2})P_oV_1$$

or

$$E_1 - E_o = (\tfrac{1}{2})(P_1 + P_o)(V_o - V_1). \tag{3.20}$$

Relationship (3.20) is the equation of Hugoniot, and will be used later in the chapter.

To investigate the shape and width of the smeared shock, it will be convenient to consider the volume function. The volume function decreases across a shock; so, for the case where we have a shock moving to the right, in the shock layer the relationship

$$\frac{\partial V}{\partial t} \leqq 0 \text{ or } \frac{dV}{dw} \geqq 0 \tag{3.21}$$

will hold. Then (3.4) can be written as

$$QV = (Ak)^2\left[\frac{dV}{dw}\right]^2. \tag{3.22}$$

From (3.15) and (3.16)

$$E - (\tfrac{1}{2})A^2V^2 = C_2 - C_1V, \tag{3.23}$$

so that by (1.38),

$$PV = \frac{\gamma - 1}{2} A^2V^2 + (\gamma - 1)C_2 - (\gamma - 1)C_1V. \tag{3.24}$$

From (3.15),

$$PV = C_1V - A^2V^2 - QV. \tag{3.25}$$

Substituting (3.25) into (3.24) and rearranging gives

$$QV = C_1V - \frac{\gamma + 1}{2} A^2V^2 - (\gamma - 1)C_2 + (\gamma - 1)C_1V$$

$$= - \frac{\gamma + 1}{2} A^2V^2 + \gamma C_1V - (\gamma - 1)C_2. \tag{3.26}$$

By considering (3.6) and the values of $V(0,t)$ and $V(x_L,t)$ given by (3.17), it is seen that the right-hand side of (3.26) is a quadratic equation in the unknown V that vanishes for $V = V_o$ and $V = V_1$. Hence, from (3.22) and (3.26),

$$(Ak)^2\left[\frac{dV}{dw}\right]^2 = QV = A^2 \frac{\gamma + 1}{2}(V_o - V)(V - V_1). \tag{3.27}$$

Now, $V_o - V$ can be rewritten as

$$\frac{V_o - V_1}{2} - \left[V - \frac{V_o + V_1}{2}\right]$$

and $V - V_1$ can be rewritten as

$$\frac{V_o - V_1}{2} + \left[V - \frac{V_o + V_1}{2}\right] ;$$

so that after dividing (3.27) through by $A^2$, we have

$$\left[k \frac{dV}{dw}\right]^2 = \frac{\gamma + 1}{2}\left\{\left[\frac{V_o - V_1}{2}\right]^2 - \left[V - \frac{V_o + V_1}{2}\right]^2\right\}, \qquad (3.28)$$

which is a quadratic in $k \frac{dV}{dw}$ with solutions

$$k \frac{dV}{dw} = \pm \left[\frac{\gamma + 1}{2}\right]^{\frac{1}{2}}\left\{\left[\frac{V_o - V_1}{2}\right]^2 - \left[V - \frac{V_o + V_1}{2}\right]^2\right\}^{\frac{1}{2}}. \qquad (3.29)$$

Dividing both sides of (3.29) by $\frac{V_o - V_1}{2}$, then making the substitution

$$V' = \frac{V - \frac{V_o + V_1}{2}}{\frac{V_o - V_1}{2}}, \qquad (3.30)$$

gives

$$k \frac{dV'}{dw} = \pm \left[\frac{\gamma + 1}{2}\right]^{\frac{1}{2}} (1 - V'^2)^{\frac{1}{2}}. \qquad (3.31)$$

Separating variables in (3.31) and integrating yields

$$W = \pm k \left[\frac{2}{\gamma + 1}\right]^{\frac{1}{2}} \int \frac{dV'}{(1 - V'^2)^{1/2}} = \pm w_o \text{ arc sin } V' + C_3 \qquad (3.32)$$

where

$$w_o = k\left[\frac{2}{\gamma + 1}\right]^{\frac{1}{2}} . \tag{3.33}$$

Finally, then, we obtain from (3.32) and (3.30) the relationship

$$V - \frac{V_o + V_1}{2} = \pm \frac{V_o - V_1}{2} \sin\left[\frac{w - C_3}{w_o}\right] ; \tag{3.34}$$

or, since $C_3$ serves only to shift the w-axis over by that amount, we may set it equal to zero to obtain, after rearranging (3.34),

$$V = \frac{V_o}{2}(1 \pm \sin \frac{w}{w_o}) + \frac{V_1}{2}(1 \mp \sin \frac{w}{w_o}). \tag{3.35}$$

Differentiating (3.35) gives

$$\frac{dV}{dw} = \pm \frac{1}{2w_o} \cos \frac{w}{w_o}\left[V_o - V_1\right] . \tag{3.36}$$

Because the shock moves to the right, $V_o > V_1$. Hence in order for the restriction that $\frac{dV}{dw} \geqq 0$ within the shock region (see equation (3.21)) to hold, let us choose the positive solution of (3.29) and restrict w such that $-\frac{1}{2}\pi w_o \leqq w \leqq \frac{1}{2}\pi w_o$.*

Suppose we examine the volume function in the interval $(-\frac{1}{2}\pi w_o, \frac{1}{2}\pi w_o)$ on the w-axis. Because of the selection of the positive root made in the above paragraph, this is

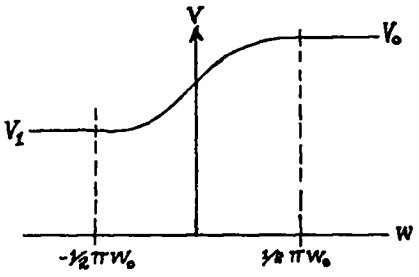


Fig. 3.1

*We could choose the negative root of (3.29) and restrict w to be within a certain interval for which $\cos\frac{w}{w_o} = 0$, but the choice made is more convenient.

$$V = \frac{V_o}{2}(1 + \sin \frac{w}{w_o}) + \frac{V_1}{2}(1 - \sin \frac{w}{w_o}). \qquad (3.37)$$

Within the interval being considered, $(-\frac{1}{2}\pi w_o, \frac{1}{2}\pi w_o)$, which is in fact the shock region, the volume function assumes the form of a smooth curve. At the left end of the interval, it has the value $V_1$ and at the right end, $V_o$. Now, by piecing (3.37) together with the particular solutions

$$V = V_o \text{ and } V = V_1 \qquad (3.38)$$

in the manner indicated by Figure 3.1, we have a continuous composite volume function with continuous first derivatives (see equation (3.36)). From (3.14), (3.15), and (3.16), it is easily seen that the velocity, pressure, and energy functions behave in a manner similar to that of the volume function in the shock region, so that they may be treated the same way. As a result, instead of a discontinuity in each of the functions at the shock, we have a continuous connection between their values in front of and behind the shock which extends over an interval of width $\pi w_o$ (see Figure 3.1 and equation (3.33)). Therefore, requirement 1 on page 50 is met.

Since the volume function is constant outside the shock region, it is easily seen from (3.4) that $Q = 0$ outside the shock region, which satisfies requirement 2 on page 50.

On page 53 (equation (3.20)) it was shown that the function values of P, V, and E, at the boundary points were related according

to the Hugoniot equation. It is seen, from (3.38) and the discussion following, that the particular solutions outside the shock layers are merely constant extensions of these values between the boundary points and the shock. Hence, requirement 3 on page 51 is satisfied.

Finally, we see that requirement 4 on page 51 is also satisfied, since we can make the shock width as small as we want to by making k small, as is demonstrated by (3.33). A tacit assumption which was made throughout the development in this chapter is logically mentioned at this point: this is the fact that the shock width is very small compared with the length of the region of flow.

We now have a system of differential equations describing gas flow which have continuous solutions. These differential equations may be used to describe the flow at each point in the gas, just as though there were no shocks at all. The shock is evident as a region over which the velocity, volume, pressure, and energy functions exhibit rapid continuous changes of the correct magnitude according to the jump conditions derived in Chapter II.

## CHAPTER V

## THE NUMERICAL TREATMENT OF THE GAS FLOW PROBLEM

For a given function $f(x,t)$ we may, in general, calculate the change in the function when x is increased by a positive amount $\Delta x$ or when t is increased by a positive amount $\Delta t$. This change in f is called the first forward difference of f with respect to x or t (according to which variable is being increased), relative to the increment $\Delta x$ or $\Delta t$. This is denoted by (in this case x is being increased)

$$\Delta_x f(x,t) = f(x + \Delta x,t) - f(x,t). \tag{4.1}$$

The symbol "$\Delta$" prefixed to a symbol representing a function is regarded in (4.1) as representing an operation performed on that function. Equation (4.1) then, can also be taken as the definition of the forward difference operator $\Delta$ .

In addition to the forward difference operator, the central difference operator $\delta$ and the backward difference operator $\nabla$ are conventionally defined by

$$\delta_x f(x,t) = f(x + \tfrac{1}{2}\Delta x,t) - f(x - \tfrac{1}{2}\Delta x,t) \tag{4.2}$$

and

$$\nabla_x f(x,t) = f(x,t) - f(x - \Delta x,t). \tag{4.3}$$

The central difference of second order is then defined by iteration according to the formula

$$\delta_x^2 f(x,t) = \delta_x \left[\delta_x f(x,t)\right]$$

$$= f(x + \Delta x,t) - 2f(x,t) + f(x - \Delta x,t). \quad (4.4)$$

An equation which relates differences of an unknown function is known as a difference equation. A difference equation arises, for example, when the derivatives in a differential equation are replaced by certain difference ratios. Since it is always possible to determine, step-by-step, successive values of the unknown function in a difference equation (provided the coefficients of the equation are sufficiently well-behaved), we have a method of obtaining approximate solutions to certain problems governed by ordinary or partial differential equations. Two important questions, which arise when the procedure described here is used, should be mentioned. These are:

1. Do both the differential equation and the corresponding difference equation possess unique solutions?
2. If the answer to question No. 1 is "yes," is the "approximate solution" obtained by solving the difference equation step-by-step really an approximation to the solution of the exact problem?

Unfortunately, complete answers to these questions are not known, and one must depend in such cases upon his knowledge of the physical problem to decide whether a function obtained by solving the difference equation is to be accepted as an approximation to the required solution.

The problem we are concerned with in this chapter is this: we want to formulate difference equations which will have, as their formal limits

as $\Delta x$ and $\Delta t$ tend to zero, the Von Neumann-Richtmyer differential equations of gas flow obtained in Chapter IV. Then after the boundary and initial conditions of whatever flow problem is being considered are specified, we will impose these, in finite difference form, on the difference equations and then proceed to obtain successive values of the hydrodynamic functions by step-by-step methods. After a solution to the difference equations is obtained in this manner, we will investigate the possibility that this is an approximation to an actual solution of the gas flow problem governed by the differential equations. The case we will consider is the one first mentioned on page 42 in Chapter III.

Now, we will proceed to write out the difference equations. Let the points of a rectangular network with spacings $\Delta x$ and $\Delta t$ be denoted by $(x_i, t_n)$ $(i = 0, 1, 2, \cdots, L;\ n = 0, 1, 2, \cdots)$. We shall also have occasion to deal with intermediate points having co-ordinates

$$x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1}),$$
$$t_{n+1/2} = \frac{1}{2}(t_n + t_{n+1}). \tag{4.5}$$

To facilitate the writing, we introduce abbreviations such as:

$$V^n_{i+1/2} = V(x_{i+1/2}, t_n),$$

etc. Central differences will be used whenever possible in the following discussion, since the error introduced is less than when forward or backward differences are used [5, p. 232].

At the point $(x_i, t_n)$, by definition of the partial derivative, (3.2) can be considered as the formal limit, as $\Delta x$ and $\Delta t$ tend to zero, of the difference equation

$$\frac{X_i^{n+1} - 2X_i^n + X_i^{n-1}}{(\Delta t)^2} = -\frac{1}{R_i^o} \frac{P_{i+\frac{1}{2}}^n + Q_{i+\frac{1}{2}}^n - P_{i-\frac{1}{2}}^n - Q_{i-\frac{1}{2}}^n}{\Delta x} \qquad (4.6)$$

A more convenient form of (4.6) is

$$X_i^{n+1} = 2X_i^n - X_i^{n-1} + \frac{(\Delta t)^2}{R_i^o \Delta x} (P_{i-\frac{1}{2}}^n + Q_{i-\frac{1}{2}}^n - P_{i+\frac{1}{2}}^n - Q_{i+\frac{1}{2}}^n). \qquad (4.7)$$

The difference equation corresponding to (3.1) is easily seen to be

$$V_{i-\frac{1}{2}}^{n+1} = \frac{1}{R_{i-\frac{1}{2}}^o} \frac{X_i^{n+1} - X_{i-1}^{n+1}}{\Delta x} . \qquad (4.8)$$

Notice that the central difference is taken around intermediate points in space. This is to keep V at the same points in the mesh as P and Q in equation (4.7) for easy application of the equation of state.

To continue, it is seen that the difference form of (3.4) is

$$Q_{i-\frac{1}{2}}^{n+1} = -\frac{\left[k R_{i-\frac{1}{2}}^o\right]^2}{V_{i-\frac{1}{2}}^{n+1}} \frac{V_{i-\frac{1}{2}}^{n+1} - V_{i-\frac{1}{2}}^n}{\Delta t} \left| \frac{V_{i-\frac{1}{2}}^{n+1} - V_{i-\frac{1}{2}}^n}{\Delta t} \right| . \qquad (4.9)$$

In obtaining (4.9), in order to keep Q at the same points in the mesh, it was necessary to use backward differences to replace $\frac{\partial V}{\partial t}$. This gives rise to a somewhat larger error, as was mentioned before, but is not serious, since Q is negligible outside the shock layers and, in any case,

artificial within. Also, before proceeding further, a specific numerical value for k should be considered. A series of experimental problems were run, using the difference equations discussed in this chapter, by Dr. Harwood G. Kolsky of the Los Alamos Scientific Laboratory using different values of k. On the basis of these experiments, the value

$$k = 1.2\,\Delta x \tag{4.10}$$

is adopted. Dr. Kolsky found that larger values of k made the changes in P, V, etc., across the shock too sluggish, while smaller values tended to give rise to large oscillations in these functions behind the shock. Since we would like to make the shock regions as narrow as possible, the value for k given by (4.10) is somewhat of a compromise, forced upon us by the limitations of numerical methods. The value of k given in (4.10) gives the shock a thickness somewhat larger than the interval $\Delta x$ (see (3.33) and later remarks on shock thickness in Chapter IV ). Substituting (4.10) into (4.9), we may write (4.9), by use of (4.8), in a different form:

$$Q_{i-\frac{1}{2}}^{n+1} = \frac{1}{V_{i-\frac{1}{2}}^{n+1}}\left[\frac{1.2}{\Delta t}\right]^2 (X_i^n - X_{i-1}^n - X_i^{n+1} + X_{i-1}^{n+1})\left|X_i^n - X_{i-1}^n - X_i^{n+1} + X_{i-1}^{n+1}\right|. \tag{4.11}$$

In order for the difference form of the expression for Q to approach (3.4) as a limit as $\Delta x$ and $\Delta t$ tend to zero, the equation (4.9) must be used; and the fact that k has $\Delta x$ as a factor, which is a consequence of experimentation, must be ignored.

In the finite difference scheme being considered, it is necessary to solve for P and E simultaneously, using the equation of state and the difference equation corresponding to (3.3), which is given by

$$\frac{E^{n+1}_{i-\frac{1}{2}} - E^{n}_{i-\frac{1}{2}}}{\Delta t} = -(P^{n+\frac{1}{2}}_{i-\frac{1}{2}} + Q^{n+\frac{1}{2}}_{i-\frac{1}{2}}) \frac{V^{n+1}_{i-\frac{1}{2}} - V^{n}_{i-\frac{1}{2}}}{\Delta t} \quad . \tag{4.12}$$

Notice that to obtain (4.12), derivatives in (3.3) were replaced by central differences taken around points which are intermediate with respect to both time and space. Again this is done to keep E at the same points in the mesh as the other thermodynamic quantities. Multiplying both sides of (4.12) by $\Delta t$ and replacing $(P^{n+\frac{1}{2}}_{i-\frac{1}{2}} + Q^{n+\frac{1}{2}}_{i-\frac{1}{2}})$ by the average value $\frac{1}{2}(P^{n}_{i-\frac{1}{2}} + Q^{n}_{i-\frac{1}{2}} + P^{n+1}_{i-\frac{1}{2}} + Q^{n+1}_{i-\frac{1}{2}})$, which gives rise to errors no larger than those already involved, we have

$$E^{n+1}_{i-\frac{1}{2}} = E^{n}_{i-\frac{1}{2}} - \tfrac{1}{2}(P^{n+1}_{i-\frac{1}{2}} + Q^{n+1}_{i-\frac{1}{2}} + P^{n}_{i-\frac{1}{2}} + Q^{n}_{i-\frac{1}{2}})(V^{n+1}_{i-\frac{1}{2}} - V^{n}_{i-\frac{1}{2}}). \tag{4.13}$$

Now, the equation of state is

$$P^{n+1}_{i-\frac{1}{2}} = \frac{E^{n+1}_{i-\frac{1}{2}}(\gamma-1)}{V^{n+1}_{i-\frac{1}{2}}} \quad . \tag{4.14}$$

If we substitute this value for $P^{n+1}_{i-\frac{1}{2}}$ into (4.13), we get

$$E^{n+1}_{i-\frac{1}{2}} = E^{n}_{i-\frac{1}{2}} - \tfrac{1}{2}\frac{E^{n+1}_{i-\frac{1}{2}}(\gamma-1)}{V^{n+1}_{i-\frac{1}{2}}}(V^{n+1}_{i-\frac{1}{2}} - V^{n}_{i-\frac{1}{2}})$$

$$- \tfrac{1}{2}(Q^{n+1}_{i-\frac{1}{2}} + P^{n}_{i-\frac{1}{2}} + Q^{n}_{i-\frac{1}{2}})(V^{n+1}_{i-\frac{1}{2}} - V^{n}_{i-\frac{1}{2}}).$$

After transposing the second term on the right-hand side and dividing through by

$$1 + \frac{\frac{1}{2}(\gamma - 1)(V^{n+1}_{i-\frac{1}{2}} - V^{n}_{i-\frac{1}{2}})}{V^{n+1}_{i-\frac{1}{2}}} ,$$

we have

$$E^{n+1}_{i-\frac{1}{2}} = \frac{E^{n}_{i-\frac{1}{2}} - \frac{1}{2}(V^{n+1}_{i-\frac{1}{2}} - V^{n}_{i-\frac{1}{2}})\,(Q^{n+1}_{i-\frac{1}{2}} + P^{n}_{i-\frac{1}{2}} + Q^{n}_{i-\frac{1}{2}})}{1 + \frac{1}{2}\,\frac{(\gamma-1)(V^{n+1}_{i-\frac{1}{2}} - V^{n}_{i-\frac{1}{2}})}{V^{n+1}_{i-\frac{1}{2}}}} . \quad (4.15)$$

After $E^{n+1}_{i-1/2}$ is determined from (4.15), $P^{n+1}_{i-1/2}$ can be found from (4.14). This completes the formulation of the difference equations.

Now that we have obtained the difference equations corresponding to the differential equations of gas flow, let us actually look at the step-by-step method of obtaining solutions to these difference equations. Suppose the quantities $X^{n}_{i}$, $X^{n-1}_{i}$, $P^{n}_{i+1/2}$, $Q^{n}_{i+1/2}$, $P^{n}_{i-1/2}$, $Q^{n}_{i-1/2}$, and $E^{n}_{i-1/2}$ are known for i = 0, 1, 2, $\cdots$, L. The position $X^{n+1}_{o}$ of the point designated by i = 0 at time $t_n + \Delta t$ is given by the boundary conditions, and the quantity $X^{n+1}_{1}$ can be found from (4.7). Then, it is easy to calculate $V^{n+1}_{1/2}$ from (4.8) and $Q^{n+1}_{1/2}$ from (4.11). The determination of $E^{n+1}_{1/2}$ and $P^{n+1}_{1/2}$ from (4.15) and (4.14) completes the "i-cycle" for i = 1, whereupon we can repeat the process for i = 2, etc. When we have finally completed the i-cycle for i = L, we say that an "n-cycle" has been completed; i.e., we now know the values of all the quantities

mentioned above at time $t_n + \Delta t$. Repetition of this process gives us values at $t_n + 2\Delta t$, $t_n + 3\Delta t$, etc.

Suppose a slight error occurs at some point in the calculation described in the above paragraph. If this error tends to increase as it is spread over the points of the finite-difference network, we say the difference equations are unstable. If, on the other hand, this small error decreases or remains the same, the equations are said to be stable. We will now discuss briefly conditions under which the stability of the equations formulated above is assured. Outside of the shock regions there is a requirement on the space-time mesh called the "Courant condition" [3, p. 237]. The mesh speed is given by

$$(c_m)^n_{i-1/2} = \frac{x^n_i - x^n_{i-1}}{\Delta t} .$$

It is the maximum speed at which a disturbance can be propagated from place to place when the finite difference scheme developed above is used. The "sound speed", c, for the conditions of pressure and volume existing in the material is the maximum speed at which a small physical disturbance propagates through the material. The Courant condition for stability may then be stated: "The mesh speed must always be greater than the sound speed." The ratio of the sound speed to the mesh speed is called the "Courant number," $\mathcal{L}$, and is given by

$$\mathcal{L} = \frac{c}{c_m} \leq 1, \qquad (4.16)$$

where the sound speed, from (2.15), can be written in difference form as

$$c^n_{i-1/2} = \left[ \frac{\gamma(P^n_{i-1/2} + Q^n_{i-1/2})}{R^n_{i-1/2}} \right]^{1/2}$$

Since the Courant condition only applies outside of shock regions, we are justified in replacing P by P + Q above. The Courant condition is not derived in this paper, so an example of the trouble that develops when this condition is violated is worthwhile.

In order to illustrate instability in the difference equations due to a violation of the Courant condition, it is convenient to consider a diagram in the (x,t)-plane similar to the one used on page 46 in Chapter III. In Figure 4.1 the paths of the points are indicated by dash lines. The velocity of a point is the reciprocal of the slope of the dash line representing its progress. Suppose a small steady disturbance is introduced into the calculation in the form of an error in the pressure term at some point. This disturbance, in the physical case, will propagate at sound speed, which is indicated by the solid line in Figure 4.1. The mesh speed is the dotted line in the figure. Notice that the point denoted by i - 1 will feel the disturbance in a correct manner at time $t_{n-1}$. At this time the pressure difference $(P^n_{i-3/2} + Q^n_{i-3/2})-(P^n_{i-1/2} + Q^n_{i-1/2})$

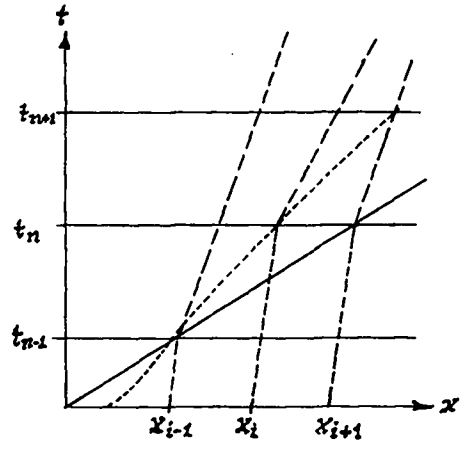

Fig. 4.1

will change slightly, altering the velocity of the point (see equation (4.6)). The point denoted by i, however, never really feels the effect of the disturbance because the disturbance passes it between points of the mesh so that its path is not altered until time $t_n$. At this time, this point feels the effect of the point denoted by i - 1 pressing close behind it in the form of a volume which is too small (see equation (4.8)). This in turn, causes the pressure $P^n_{i-1/2}$ to be too large (see equation (4.14)). As a result, the velocity of the point denoted by i may increase by too large an amount, which in extreme cases may cause the paths of the points denoted by i and i + 1 to cross, which quickly makes gibberish of the entire calculation.

A stable situation is demonstrated in Figure 4.2*. Here the disturbance propagates at exactly mesh speed, so that each point's velocity is altered correctly.

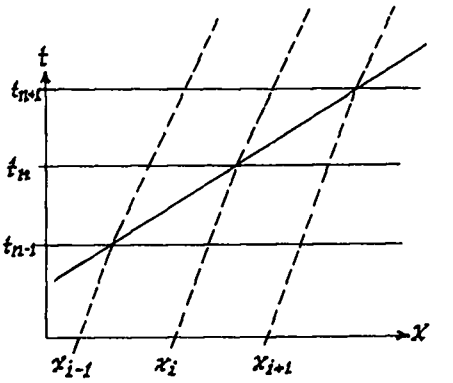

Fig. 4.2

Comparison of the stable and unstable situations discussed above demonstrates that reducing the time interval by a sufficient amount will correct an unstable situation caused by a violation of the Courant condition.

Another stability condition, which applies only inside the shock layers, can be derived from expressions given by Von Neumann and

*Notice that, in this case, the time interval is smaller than it was for the unstable case.

Richtmyer in their paper, but is merely written down here. This is:

$$\frac{V^{n-1}_{i+1/2} - V^{n}_{i+1/2}}{V^{n}_{i+1/2}} < \frac{1}{5.76} \tag{4.17}$$

The number 5.76 is related to the shock width, since in the derivation it arises as four times the square of the number 1.2 which appears in (4.10).

Recently Dr. George N. White, Jr., of the Los Alamos Scientific Laboratory, has developed a stability condition which effectively combines (4.16) and (4.17), making the stability condition calculation much quicker and easier [6].

Thus, we have a system of difference equations which we hope will give a good approximation to a solution of the gas flow problem when the initial and boundary conditions are given. Let us proceed to a numerical example to investigate this possibiliby.

We have already indicated (page 65) the method of obtaining successive values in time for the hydrodynamic functions at a multitude of points in the gas; namely, by stepwise numerical procedures applied to the difference equations formulated above. Now let us consider actual numerical values for the initial and boundary conditions, and see if results obtained by applying this method appear to approximate the actual solution to the gas flow problem.

For the gas used in the following numerical work, $\gamma = 2$. Let the initial conditions be

$$P(x_n, 0) = P_o = V_o = E_o = R_o = 1, \; U_o = 0 \tag{4.18}$$

at all points in the gas.

Before proceeding, it is important that the $(\Delta x, \Delta t)$-mesh to be used be determined. Application of the Courant condition, using the values of the thermodynamic functions specified in the initial conditions, gives

$$1 \geq \frac{\Delta t}{\Delta x}\left[\frac{\gamma P}{R}\right]^{1/2} = \sqrt{2}\,\frac{\Delta t}{\Delta x}\,;$$

so that the mesh to be used must be such that

$$\Delta x \geq \sqrt{2}\,\Delta t. \tag{4.19}$$

After the calculation is started, the conditions given by (4.19) will change somewhat. In order to avoid changing the mesh* in the event of a Courant condition violation (or a violation of (4.17)) it is a good idea to choose the relationship between $\Delta x$ and $\Delta t$ such that there is some assurance the stability conditions will be satisfied during the entire calculation. With this in mind the values

$$\Delta x = 1.3, \;\; \Delta t = .125, \tag{4.20}$$

were chosen.

To continue, let the boundary conditions be

$$\begin{aligned} U(0,t) &= .26 \text{ space units per unit time,} \\ U(x_L,t) &= \;\; 0 \text{ space units per unit time.} \end{aligned} \tag{4.21}$$

We have already seen (pages 42-44) that it is possible to obtain the analytic solution to the flow problem for initial conditions and

* In practice this is quite easy to do but involves a somewhat longer machine code.

boundary conditions of the type given by (4.18) and (4.21), respectively. Let us consider now the actual numerical values of these solutions, so that we may compare them with the numerical results to be obtained. When the initial conditions (4.18) and the boundary conditions (4.21) are applied, the numerical values of these solutions are quite easily obtained by the method outlined in Chapter III on page 44. For example, when this method is used, the velocity function behind the shock is found to be

$$U_1 = .26; \tag{4.22}$$

then, (2.61) becomes

$$U^2 - .39U - 2 = 0. \tag{4.23}$$

The positive solution of (4.23) is the incident shock velocity and is approximately (to eight significant figures)

$$U_+ = 1.62259412 . \tag{4.24}$$

From (2.35), using the value of $U_+$ given by (4.24),

$$R_1 = 1.19081250 . \tag{4.25}$$

Also, from (2.36), using the values given by (4.24) and (4.25),

$$P_1 = 1.42187447 . \tag{4.26}$$

From the equation of state, (4.25), and (4.26),

$$E_1 = 1.19403723. \tag{4.27}$$

Substituting the numerical values given by (4.18), (4.22), (4.25), (4.26), and (4.27) into (2.65) —(2.69) yields the desired analytic solution to the differential equations of gas flow. The value of the pressure function behind the shock after it has been reflected is quickly found, by

use of (2.75), to be

$$P_2 = 1.96449733; \tag{4.28}$$

and the reflected shock velocity is the negative root of the equation

$$U^2 - .13U - 2.42187447 = 0, \tag{4.29}$$

which is obtained by substituting the numerical values of $U_1$ and $c_1$ into (2.70). Hence

$$U_- = - 1.49259413. \tag{4.30}$$

The difference equations developed earlier in this chapter with the initial conditions (4.18) and boundary conditions (4.21) imposed were solved in the manner outlined on page 65. Usually one would not attempt calculations of this type by hand, especially when there is a large number of points (spacewise)involved, because of the great amount of timerequired plus the high probability of human errors occurring. Such a calculation is quite feasible, however,when there is a high-speed digital calculating machine available. The machines presently available are capable of carrying out long, intricate, and error-free numerical calculations quite automatically and at very high speeds [8]. Such a machine, the IBM Type 701 EDPM, was used in carrying out the actual calculations indicated on page 65.

In this particular calculation, fifty "mass points", i.e., points on the space scale separated by $\Delta x$ and hence carrying a mass of $R_i^o \Delta x$ (since R is initially constant) evenly spaced at 1.3 units apart were

used. The calculation was started by moving the first mass point as a piston, .0325 space units per time cycle, according to (4.20) and (4.22). The shock immediately formed and moved into the gas with a velocity

$$(U_+)_N = 1.648 \pm .0046. \qquad (4.31)$$

The parenthesis with subscript "N" indicates that the value of the quantity was arrived at numerically. The value for the shock velocity given by (4.31) was actually obtained from Figure 4.3, which shows the shock quite clearly, at two different times, as a rapid change in the fluid velocity. This value was found by following a point at which the fluid velocity is constant (for this particular case, the value was chosen to be u = .12) a trick that was used on page 32 in Chapter III. The velocity of the reflected shock obtained in a similar manner from Figure 4.4 is (the value used here was p = 1.7)

$$(U_-)_N = - 1.485 \pm .0047. \qquad (4.32)$$

Table I gives the values of all the hydrodynamic functions at t = 13.75 (after 110 n-cycles) before the shock is reflected. Table II is similar to Table I, but for a later time t = 63.75 after the shock has been reflected.* Notice that the lower part of Figure 4.3 agrees timewise with Table I, and that the lower part of Figure 4.4 agrees timewise with Table II. Notice that there are slight oscillations in all the functions directly behind the shock: this is because

*The approximate position of the shock can be found by locating the maximum value of Q.

of the limitations which are inherent in any numerical method and is not serious, as they settle down to the correct value quite rapidly.

After the shock was reflected, the value of the reflected pressure ratio,

$$\left[\frac{P_2}{P_1}\right]_N = 1.3833, \tag{4.33}$$

was computed directly from the listing printed by the machine.

Using the numerical values in Table I, it is easily verified that the important relationships in Chapter III are all satisfied. The incident and reflected shock velocities given by (4.31) and (4.32) agree with the theoretical values given by (4.24) and (4.30) within 0.4 per cent; the reflected pressure ratios given by (4.33) and by the combination of (4.28) and (4.26) agree within 0.2 per cent. The values of the hydrodynamic functions listed in Table I do not attain this degree of agreement with the values indicated by (4.22), (4.25), (4.26), and (4.27) until some time after the shock has passed, because of the oscillations right behind the shock which were mentioned earlier.

In conclusion, by the agreement between the theoretical and calculated values of the important quantities demonstrated above, it is clear that the solution to the system of difference equations developed earlier in this chapter is quite a good approximation to the solution to the differential equations of gas flow when both have the same initial and boundary conditions given. The same method can be applied to other, more complicated geometries of flow and also to higher dimensions [7].

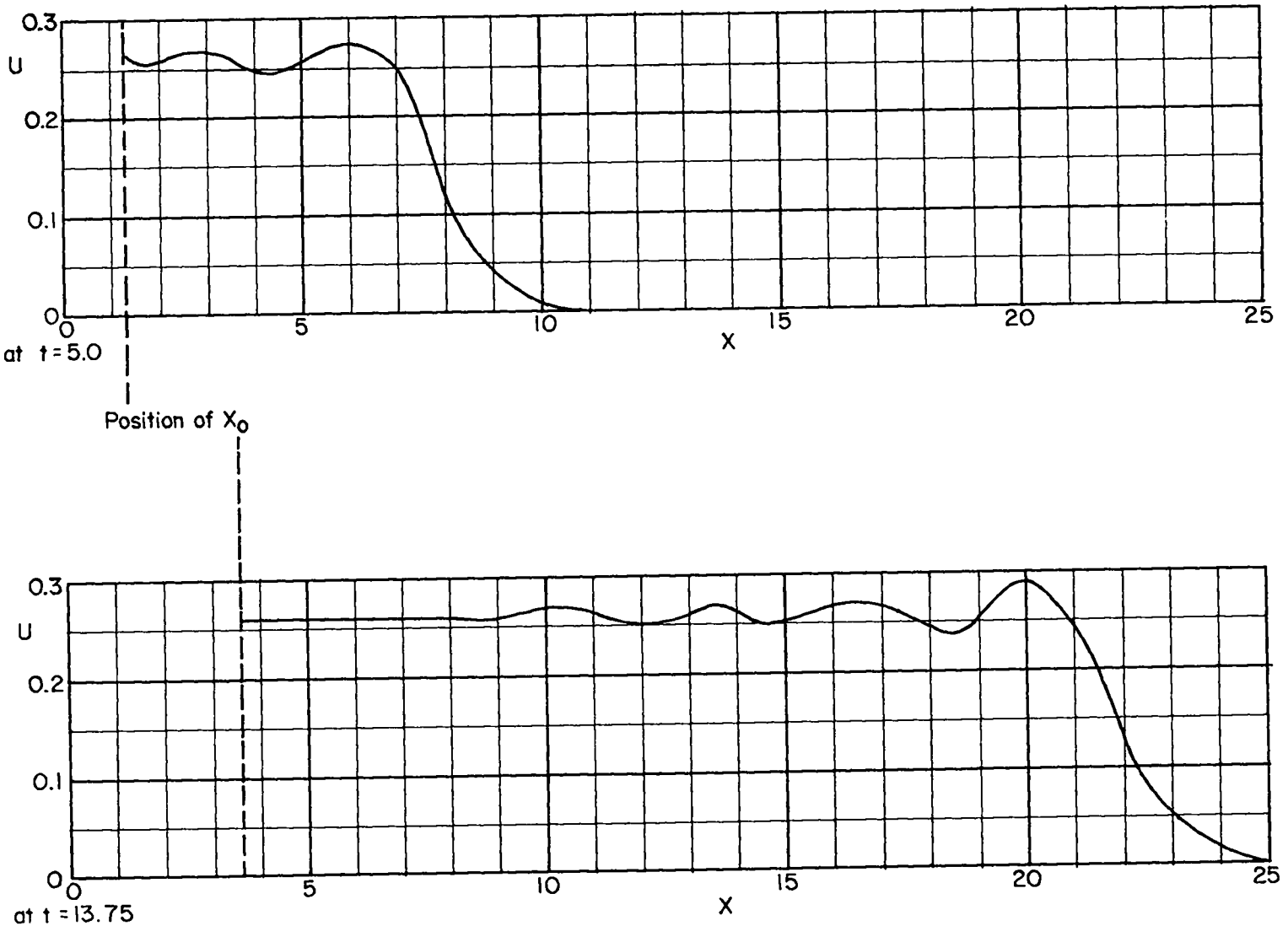


Fig. 4.3. Comparison of Fluid Velocity at two different times before reflection of the shock.

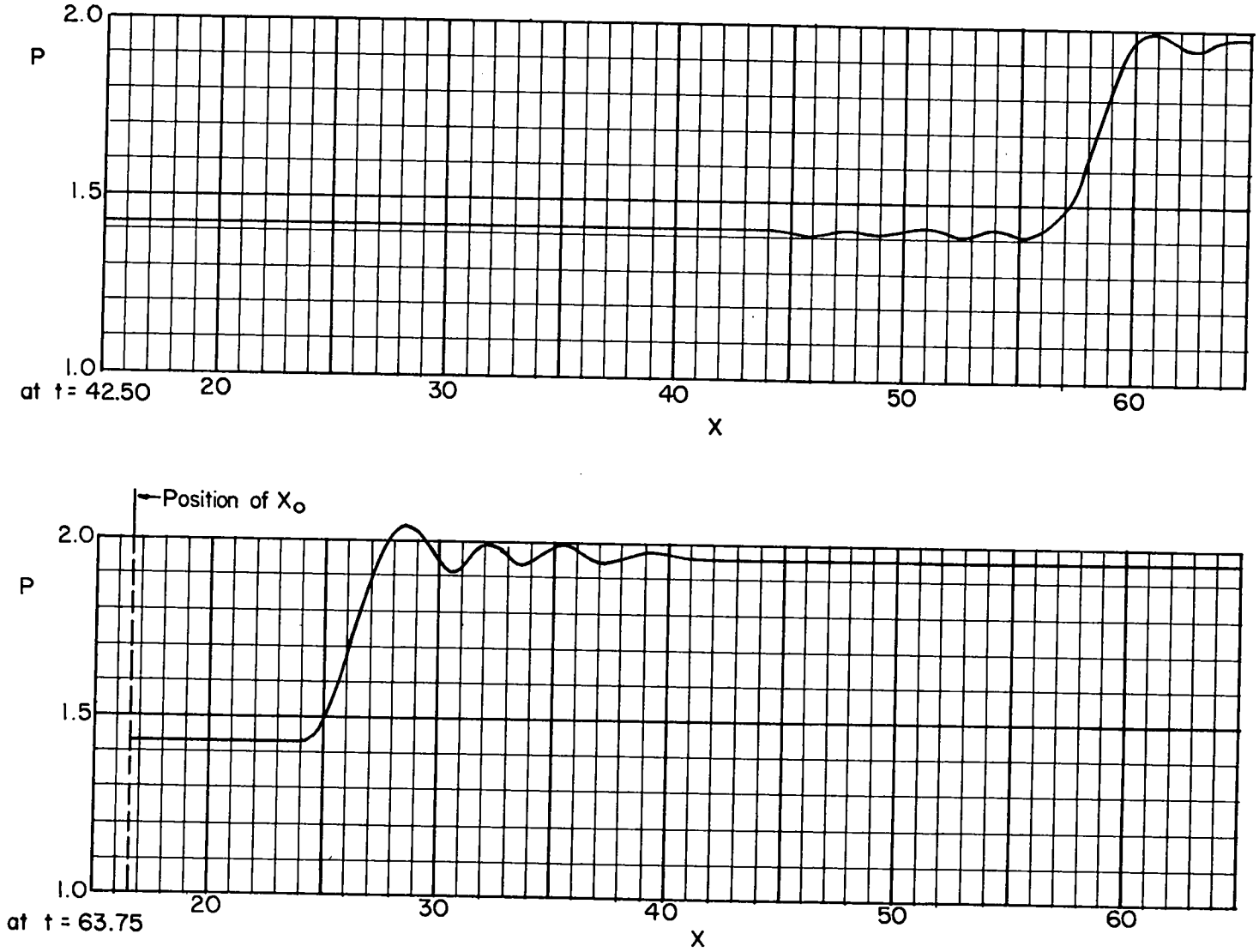


Fig. 4.4. Pressure at two different times after reflection of the shock.

TABLE I

VALUES OF THE HYDRODYNAMIC FUNCTIONS AT t = 13.75 OBTAINED BY NUMERICAL METHODS

| i | $E_{i+\frac{1}{2}}$ | $(P + Q)_{i+\frac{1}{2}}$ | $Q_{i+\frac{1}{2}}$ | $X_i$ | $U_i$ | $V_{i+\frac{1}{2}}$ |
|---|---|---|---|---|---|---|
| 0 | 1.1973 | 1.4211 | 0.0000 | 3.5750 | 0.2600 | 0.8425 |
| 1 | 1.1955 | 1.4240 | 0.0000 | 4.6702 | 0.2601 | 0.8396 |
| 2 | 1.1929 | 1.4185 | 0.0000 | 5.7617 | 0.2601 | 0.8410 |
| 3 | 1.1959 | 1.4258 | 0.0000 | 6.8550 | 0.2592 | 0.8387 |
| 4 | 1.1928 | 1.4188 | 0.0001 | 7.9453 | 0.2621 | 0.8408 |
| 5 | 1.1939 | 1.4213 | - 0.0001 | 9.0383 | 0.2562 | 0.8399 |
| 6 | 1.1966 | 1.4281 | 0.0001 | 10.1302 | 0.2649 | 0.8380 |
| 7 | 1.1891 | 1.4100 | - 0.0001 | 11.2196 | 0.2560 | 0.8433 |
| 8 | 1.1989 | 1.4334 | - 0.0001 | 12.3159 | 0.2596 | 0.8364 |
| 9 | 1.1936 | 1.4214 | 0.0005 | 13.4031 | 0.2668 | 0.8400 |
| 10 | 1.1867 | 1.4043 | - 0.0003 | 14.4952 | 0.2499 | 0.8449 |
| 11 | 1.2036 | 1.4447 | - 0.0002 | 15.5936 | 0.2626 | 0.8330 |
| 12 | 1.1946 | 1.4248 | 0.0016 | 16.6765 | 0.2725 | 0.8393 |
| 13 | 1.1768 | 1.3811 | - 0.0002 | 17.7676 | 0.2424 | 0.8520 |
| 14 | 1.2052 | 1.4455 | - 0.0033 | 18.8752 | 0.2520 | 0.8382 |
| 15 | 1.2153 | 1.4778 | - 0.0045 | 19.9566 | 0.2955 | 0.8249 |
| 16 | 1.1362 | 1.3155 | 0.0266 | 21.0290 | 0.2447 | 0.8816 |
| 17 | 1.0481 | 1.1096 | 0.0114 | 22.1751 | 0.1171 | 0.9543 |
| 18 | 1.0103 | 1.0216 | 0.0009 | 23.4157 | 0.0304 | 0.9898 |
| 19 | 1.0016 | 1.0033 | 0.0000 | 24.7042 | 0.0052 | 0.9984 |
| 20 | 1.0002 | 1.0004 | 0.0000 | 26.0003 | 0.0007 | 0.9998 |
| 21 | 1.0000 | 1.0000 | 0.0000 | 27.3000 | 0.0000 | 1.0000 |
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| 48 | 1.0000 | 1.0000 | 0.0000 | 62.4000 | 0.0000 | 1.0000 |
| 49 | | | 0.0000 | 63.7000 | 0.0000 | |

TABLE II

VALUES OF THE HYDRODYNAMIC FUNCTIONS AT t = 63.75 OBTAINED BY NUMERICAL METHODS

| i | $E_{i+\frac{1}{2}}$ | $(P + Q)_{i+\frac{1}{2}}$ | $Q_{i+\frac{1}{2}}$ | $X_i$ | $U_i$ | $V_{i+\frac{1}{2}}$ |
|---|---|---|---|---|---|---|
| 0 | 1.1976 | 1.4219 | 0.0000 | 16.5750 | 0.2600 | 0.8422 |
| 1 | 1.1946 | 1.4218 | 0.0000 | 17.6699 | 0.2600 | 0.8402 |
| 2 | 1.1944 | 1.4220 | 0.0000 | 18.7622 | 0.2599 | 0.8399 |
| 3 | 1.1941 | 1.4217 | 0.0000 | 19.8541 | 0.2601 | 0.8399 |
| 4 | 1.1942 | 1.4221 | 0.0000 | 20.9461 | 0.2598 | 0.8398 |
| 5 | 1.1941 | 1.4219 | 0.0000 | 22.0377 | 0.2602 | 0.8398 |
| 6 | 1.1946 | 1.4232 | 0.0000 | 23.1295 | 0.2595 | 0.8394 |
| 7 | 1.1968 | 1.4286 | 0.0001 | 24.2207 | 0.2591 | 0.8378 |
| 8 | 1.2092 | 1.4597 | 0.0015 | 25.3098 | 0.2522 | 0.8292 |
| 9 | 1.2575 | 1.5929 | 0.0162 | 26.3878 | 0.2229 | 0.7975 |
| 10 | 1.3575 | 1.8657 | 0.0304 | 27.4246 | 0.1282 | 0.7396 |
| 11 | 1.4343 | 2.0510 | 0.0032 | 28.3861 | 0.0320 | 0.7004 |
| 12 | 1.4128 | 1.9817 | -0.0049 | 29.2967 | -0.0364 | 0.7111 |
| 13 | 1.3811 | 1.8982 | -0.0003 | 30.2212 | 0.0128 | 0.7275 |
| 14 | 1.4018 | 1.9585 | 0.0027 | 31.1669 | 0.0250 | 0.7167 |
| 15 | 1.4232 | 2.0134 | 0.0000 | 32.0987 | -0.0120 | 0.7064 |
| 16 | 1.4033 | 1.9582 | -0.0016 | 33.0170 | -0.0154 | 0.7160 |
| 17 | 1.3910 | 1.9256 | 0.0000 | 33.9479 | 0.0130 | 0.7223 |
| 18 | 1.4075 | 1.9725 | 0.0009 | 34.8869 | 0.0107 | 0.7139 |
| 19 | 1.4157 | 1.9945 | 0.0000 | 35.8149 | -0.0106 | 0.7098 |
| 20 | 1.4012 | 1.9535 | -0.0005 | 36.7377 | -0.0071 | 0.7171 |
| 21 | 1.3973 | 1.9433 | 0.0000 | 37.6699 | 0.0091 | 0.7191 |
| 22 | 1.4079 | 1.9729 | 0.0003 | 38.6047 | 0.0052 | 0.7137 |
| 23 | 1.4099 | 1.9784 | 0.0000 | 39.5325 | -0.0070 | 0.7127 |
| 24 | 1.4033 | 1.9598 | -0.0001 | 40.4590 | -0.0029 | 0.7160 |

TABLE II (Continued)

| i | $E_{i+\frac{1}{2}}$ | $(P + Q)_{i+\frac{1}{2}}$ | $Q_{i+\frac{1}{2}}$ | $X_i$ | $U_i$ | $V_{i+\frac{1}{2}}$ |
|---|---|---|---|---|---|---|
| 25 | 1.4015 | 1.9548 | 0.0000 | 41.3898 | 0.0045 | 0.7170 |
| 26 | 1.4053 | 1.9655 | 0.0000 | 42.3218 | 0.0019 | 0.7150 |
| 27 | 1.4077 | 1.9722 | 0.0000 | 43.2513 | -0.0018 | 0.7138 |
| 28 | 1.4052 | 1.9651 | 0.0000 | 44.1792 | -0.0024 | 0.7151 |
| 29 | 1.4024 | 1.9574 | 0.0000 | 45.1088 | 0.0007 | 0.7164 |
| 30 | 1.4047 | 1.9640 | 0.0000 | 46.0402 | 0.0035 | 0.7153 |
| 31 | 1.4074 | 1.9712 | 0.0000 | 46.9700 | -0.0016 | 0.7140 |
| 32 | 1.4046 | 1.9638 | 0.0000 | 47.8982 | -0.0030 | 0.7152 |
| 33 | 1.4032 | 1.9594 | 0.0000 | 48.8280 | 0.0027 | 0.7161 |
| 34 | 1.4050 | 1.9650 | 0.0000 | 49.7589 | 0.0009 | 0.7150 |
| 35 | 1.4063 | 1.9685 | 0.0000 | 50.6884 | -0.0018 | 0.7144 |
| 36 | 1.4050 | 1.9648 | 0.0000 | 52.5162 | -0.0001 | 0.7151 |
| 37 | 1.4029 | 1.9593 | 0.0000 | 52.5468 | 0.0006 | 0.7160 |
| 38 | 1.4057 | 1.9666 | 0.0000 | 53.4776 | 0.0008 | 0.7148 |
| 39 | 1.4062 | 1.9686 | 0.0000 | 54.4068 | -0.0009 | 0.7143 |
| 40 | 1.4033 | 1.9598 | 0.0000 | 55.3354 | -0.0007 | 0.7160 |
| 41 | 1.4045 | 1.9643 | 0.0000 | 56.2662 | 0.0008 | 0.7150 |
| 42 | 1.4063 | 1.9680 | 0.0000 | 57.1957 | 0.0006 | 0.7146 |
| 43 | 1.4040 | 1.9632 | 0.0000 | 58.1247 | -0.0011 | 0.7151 |
| 44 | 1.4043 | 1.9623 | 0.0000 | 59.0544 | -0.0003 | 0.7157 |
| 45 | 1.4054 | 1.9676 | 0.0000 | 59.9848 | 0.0016 | 0.7143 |
| 46 | 1.4052 | 1.9640 | 0.0000 | 60.9133 | -0.0010 | 0.7155 |
| 47 | 1.4034 | 1.9620 | 0.0000 | 61.8435 | -0.0010 | 0.7153 |
| 48 | 1.4140 | 1.9669 | 0.0000 | 62.7733 | 0.0017 | 0.7128 |
| 49 | | | 0.0000 | 63.7000 | 0.0000 | |

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